

Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension

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Abstract

For the L^2 supercritical generalized Korteweg-de Vries equation, we proved in [2] the existence and uniqueness of an N -parameter family of N -solitons. Recall that, for any N given solitons, we call N -soliton a solution of the equation which behaves as the sum of these N solitons asymptotically as $t \rightarrow +\infty$. In the present paper, we also construct an N -parameter family of N -solitons for the supercritical nonlinear Schrödinger equation, in dimension 1 for the sake of simplicity. Nevertheless, we do not obtain any classification result; but recall that, even in subcritical and critical cases, no general uniqueness result has been proved yet.

1 Introduction

1.1 The nonlinear Schrödinger equation

We consider the L^2 supercritical focusing nonlinear Schrödinger equation in one dimension:

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases} \quad (\text{NLS})$$

where $(t, x) \in \mathbb{R}^2$, $p > 5$ is real, and u is a complex-valued function. Recall first that Ginibre and Velo [6] proved that (NLS) is locally well-posed in $H^1(\mathbb{R})$ for $p > 1$: for any $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a unique maximal solution $u \in C([0, T], H^1(\mathbb{R}))$ of (NLS). Moreover, either $T = +\infty$ or $T < +\infty$ and then $\lim_{t \rightarrow T} \|\partial_x u(t)\|_{L^2} = +\infty$. It is also well-known that H^1 solutions of (NLS) satisfy the following three conservation laws: for all $t \in [0, T]$,

$$\begin{aligned} M(u(t)) &= \int |u(t)|^2 = M(u_0) \quad (\text{mass}), \\ E(u(t)) &= \frac{1}{2} \int |\partial_x u(t)|^2 - \frac{1}{p+1} \int |u(t)|^{p+1} = E(u_0) \quad (\text{energy}), \\ P(u(t)) &= \text{Im} \int \partial_x u(t) \bar{u}(t) = P(u_0) \quad (\text{momentum}). \end{aligned}$$

Recall also that (NLS) admits the following symmetries.

- Space-time translation invariance: if $u(t, x)$ satisfies (NLS), then for any $t_0, x_0 \in \mathbb{R}$, $w(t, x) = u(t - t_0, x - x_0)$ also satisfies (NLS).
- Scaling invariance: if $u(t, x)$ satisfies (NLS), then for any $\lambda > 0$, $w(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ also satisfies (NLS).

- Phase invariance: if $u(t, x)$ satisfies (NLS), then for any $\gamma_0 \in \mathbb{R}$, $w(t, x) = u(t, x)e^{i\gamma_0}$ also satisfies (NLS).
- Galilean invariance: if $u(t, x)$ satisfies (NLS), then for any $v_0 \in \mathbb{R}$, $w(t, x) = u(t, x - v_0 t)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t)}$ also satisfies (NLS).

We now consider solitary waves of (NLS), in other words solutions of the form $u(t, x) = e^{ic_0 t} Q_{c_0}(x)$, where $c_0 > 0$ and Q_{c_0} is solution of

$$Q_{c_0} > 0, \quad Q_{c_0} \in H^1(\mathbb{R}), \quad Q_{c_0}'' + Q_{c_0}^p = c_0 Q_{c_0}. \quad (1.1)$$

Recall that such positive solution of (1.1) exists and is unique up to translations, and is moreover the solution of a variational problem: we call Q_{c_0} the solution of (1.1) which is even, and we denote $Q := Q_1$. By the symmetries of (NLS), for any $\gamma_0, v_0, x_0 \in \mathbb{R}$,

$$R_{c_0, \gamma_0, v_0, x_0}(t, x) = Q_{c_0}(x - v_0 t - x_0)e^{i(\frac{v_0}{2}x - \frac{v_0^2}{4}t + c_0 t + \gamma_0)}$$

is also a solitary wave of (NLS), moving on the line $x = v_0 t + x_0$, that we also call *soliton*.

Finally recall that, in the supercritical case $p > 5$, solitons are *unstable* (see [8]). A striking illustration of this fact is the following result of Duyckaerts and Roudenko [5] (adapted from a previous work of Duyckaerts and Merle [4]), obtained for the 3d focusing cubic nonlinear Schrödinger equation (NLS-3d), which is also L^2 supercritical and H^1 subcritical as in our case.

Proposition 1.1 ([5]). *Let $A \in \mathbb{R}$. If $t_0 = t_0(A) > 0$ is large enough, then there exists a radial solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of (NLS-3d) such that*

$$\forall b \in \mathbb{R}, \exists C > 0, \forall t \geq t_0, \quad \|U^A(t) - e^{it}Q - Ae^{(i-e_0)t}Y^+\|_{H^b} \leq Ce^{-2e_0 t},$$

where $e_0 > 0$ and $Y^+ \neq 0$ is in the Schwartz space \mathcal{S} .

In particular, $U^A(t) \neq e^{it}Q$ if $A \neq 0$, whereas $\lim_{t \rightarrow +\infty} \|U^A(t) - e^{it}Q\|_{H^1} = 0$. Note that, in the subcritical and critical cases $p \leq 5$, no such special solutions $U^A(t)$ can exist, due to a variational characterization of Q . Indeed, if $\lim_{t \rightarrow +\infty} \|u(t) - e^{it}Q\|_{H^1} = 0$, then $u(t) = e^{it}Q$ in this case. The purpose of this paper is to extend Proposition 1.1 to multi-solitons.

1.2 Multi-solitons

Now, we focus on multi-soliton solutions. Given $4N$ parameters defining $N \geq 2$ solitons with different speeds,

$$v_1 < \dots < v_N, \quad c_1, \dots, c_N \in \mathbb{R}_+^*, \quad \gamma_1, \dots, \gamma_N \in \mathbb{R}, \quad x_1, \dots, x_N \in \mathbb{R}, \quad (1.2)$$

we set

$$R_j(t) = R_{c_j, \gamma_j, v_j, x_j}(t) \quad \text{and} \quad R(t) = \sum_{j=1}^N R_j(t),$$

and we call N -soliton a solution $u(t)$ of (NLS) such that

$$\|u(t) - R(t)\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Let us recall known results on multi-solitons.

- In the L^2 subcritical and critical cases, *i.e.* for (NLS) with $p \leq 5$, there exists a large literature on the problem of existence of multi-solitons and on their properties. Merle [12] first established an existence result in the critical case, as a consequence of a blow up result and the conformal invariance. This result was extended by Martel and Merle [10] to the subcritical case, using arguments developed by Martel, Merle and Tsai [11] for the stability in H^1 of solitons. Nevertheless, we recall that no general uniqueness result has been proved, contrarily to the generalized Korteweg-de Vries (gKdV) equation (see [9]).

For other stability and asymptotic stability results on multi-solitons of some nonlinear Schrödinger equations, see [13, 14, 15].

- In the L^2 supercritical case, *i.e.* in a situation where solitons are known to be unstable, Côte, Martel and Merle [3] have recently proved the existence of at least one multi-soliton solution for (NLS):

Theorem 1.2 ([3]). *Let $p > 5$ and $N \geq 2$. Let $v_1 < \dots < v_N$, $(c_1, \dots, c_N) \in (\mathbb{R}_+^*)^N$, $(\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ and $(x_1, \dots, x_N) \in \mathbb{R}^N$. There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $\varphi \in C([T_0, +\infty), H^1)$ of (NLS) such that*

$$\forall t \in [T_0, +\infty), \quad \|\varphi(t) - R(t)\|_{H^1} \leq C e^{-\sigma_0^{3/2} t}.$$

Recall that, with respect to [10, 11], the proof of Theorem 1.2 relies on an additional topological argument to control the unstable nature of the solitons. Finally, recall that Theorem 1.2 was also obtained for the L^2 supercritical gKdV equation, and has been a crucial starting point in [2] to obtain the multi-existence and the classification of multi-solitons. It is a similar multi-existence result that we propose to prove in this paper.

1.3 Main result and outline of the paper

The whole paper is devoted to prove the following theorem of existence of a family of multi-solitons for the supercritical (NLS) equation.

Theorem 1.3. *Let $p > 5$, $N \geq 2$, $v_1 < \dots < v_N$, $(c_1, \dots, c_N) \in (\mathbb{R}_+^*)^N$, $(\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ and $(x_1, \dots, x_N) \in \mathbb{R}^N$. Denote $R = \sum_{j=1}^N R_{c_j, \gamma_j, v_j, x_j}$.*

Then there exist $\gamma > 0$ and an N -parameter family $(\varphi_{A_1, \dots, A_N})_{(A_1, \dots, A_N) \in \mathbb{R}^N}$ of solutions of (NLS) such that, for all $(A_1, \dots, A_N) \in \mathbb{R}^N$, there exist $C > 0$ and $t_0 > 0$ such that

$$\forall t \geq t_0, \quad \|\varphi_{A_1, \dots, A_N}(t) - R(t)\|_{H^1} \leq C e^{-\gamma t},$$

and if $(A'_1, \dots, A'_N) \neq (A_1, \dots, A_N)$, then $\varphi_{A'_1, \dots, A'_N} \neq \varphi_{A_1, \dots, A_N}$.

Remark 1.4. As underlined above, the question of the classification of multi-solitons is open for the (NLS) equation, even in the subcritical case, while it was obtained in [2] for the supercritical gKdV equation, and in [9] for the subcritical and critical cases. Although we expect that the family constructed in Theorem 1.3 characterizes all multi-solitons, the lack of monotonicity properties such as for the gKdV equation does not allow to prove it for now.

The paper is organized as follows. In the next section, we briefly recall some well-known results on multi-solitons and on the linearized equation. One of the most important facts about the linearized equation, also strongly used in [5, 3], is the determination of the spectrum of the linearized operator \mathcal{L} around the soliton $e^{it}Q$ (proved in [16] and [7]): $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover e_0 and $-e_0$ are simple eigenvalues of \mathcal{L} with eigenfunctions Y^+ and Y^- . Indeed, Y^\pm allow to control the negative directions of the linearized energy around a soliton (see Proposition 2.4). Moreover, by a simple scaling argument, we determine the eigenvalues of the linearized operator around $e^{ic_j t}Q_{c_j}$, and in particular $\pm e_j = \pm c_j^{3/2} e_0$ are simple eigenvalues with eigenfunctions Y_j^\pm (see Notation 2.7 for precise definitions).

In Section 3, we construct the family $(\varphi_{A_1, \dots, A_N})$ described in Theorem 1.3. To do this, we first claim Proposition 3.1, which is the key point of the proof of the multi-existence result as in [2], and can be summarized as follows. *Let φ be a multi-soliton given by Theorem 1.2, $j \in \llbracket 1, N \rrbracket$ and $A_j \in \mathbb{R}$. Then there exists a solution $u(t)$ of (NLS) such that*

$$\|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t},$$

for t large and for some small $\gamma > 0$. This means that, similarly as in [5] for one soliton, we can perturb the multi-soliton φ locally around one given soliton at the order $e^{-e_j t}$. Since it is not significant to perturb φ at order e_j before order e_k if $e_j > e_k$, the construction of $\varphi_{A_1, \dots, A_N}$ has to be done following values (possibly equal) of e_j .

Finally, to prove Proposition 3.1, we follow the strategy of the proof of the similar proposition in [2], except for the monotonicity property of the energy which does not hold for the (NLS) equation. If this property of monotonicity was necessary to obtain the classification, we prove that a slightly different functional estimated regardless its sign is sufficient to reach our purpose. We also rely on refinements of arguments developed in [3], in particular the topological argument to control the unstable directions.

2 Preliminary results

Notation 2.1. They are available in the whole paper.

- (a) We denote $\partial_x v = v_x$ the partial derivative of v with respect to x .
- (b) For $h \in \mathbb{C}$, we denote $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$.
- (c) For $f, g \in L^2$, $(f, g) = \operatorname{Re} \int f \bar{g}$ denotes the real scalar product.
- (d) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1 + \xi^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R})\}$, and in particular $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.
- (e) If a and b are two functions of t and if b is positive, we write $a = O(b)$ when there exists a constant $C > 0$ independent of t such that $|a(t)| \leq Cb(t)$ for all t .

2.1 Linearized operator around a stationary soliton

The linearized equation appears if one considers a solution of (NLS) close to the soliton $e^{it}Q$. More precisely, if $u(t, x) = e^{it}(Q(x) + h(t, x))$ satisfies (NLS), then h satisfies $\partial_t h + \mathcal{L}h = O(h^2)$, where the operator \mathcal{L} is defined for $v = v_1 + iv_2$ by

$$\mathcal{L}v = -L_-v_2 + iL_+v_1,$$

and the self-adjoint operators L_+ and L_- are defined by

$$L_+v_1 = -\partial_x^2 v_1 + v_1 - pQ^{p-1}v_1, \quad L_-v_2 = -\partial_x^2 v_2 + v_2 - Q^{p-1}v_2.$$

The spectral properties of \mathcal{L} are well-known (see [7, 16] for instance), and summed up in the following proposition.

Proposition 2.2 ([7, 16]). *Let $\sigma(\mathcal{L})$ be the spectrum of the operator \mathcal{L} defined on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}) = \{i\xi \ ; \ \xi \in \mathbb{R}, |\xi| \geq 1\}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\} \quad \text{with } e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of \mathcal{L} with eigenfunctions Y^+ and $Y^- = \overline{Y^+}$ which have an exponential decay at infinity. Finally, the null space of \mathcal{L} is spanned by $\partial_x Q$ and iQ , and as a consequence, the null space of L_+ is spanned by $\partial_x Q$ and the null space of L_- is spanned by Q .

Remark 2.3. By standard ODE techniques, we can quantify the exponential decay of Y^\pm and $\partial_x Y^\pm$ at infinity. In fact, there exist $\eta_0 > 0$ and $C > 0$ such that, for all $x \in \mathbb{R}$,

$$|Y^\pm(x)| + |\partial_x Y^\pm(x)| \leq Ce^{-\eta_0|x|}.$$

Moreover, \mathcal{L} , L_+ and L_- satisfy some properties of positivity or coercivity. The following proposition sums up the two properties useful for our purpose. Note that the first one is proved in [16], while the second one is proved in [4, 5].

Proposition 2.4 ([16, 5]). *(i) For all $f \in H^1 \setminus \{\lambda Q \ ; \ \lambda \in \mathbb{R}\}$ real-valued, one has $\int (L_- f) f > 0$.*

(ii) There exists $\kappa_0 > 0$ such that, for all $v = v_1 + iv_2 \in H^1$,

$$(L_+v_1, v_1) + (L_-v_2, v_2) \geq \frac{1}{\kappa_0} \|v\|_{H^1}^2 - \kappa_0 \left[\left(\int \partial_x Q v_1 \right)^2 + \left(\int Q v_2 \right)^2 + \left(\operatorname{Im} \int Y^+ \bar{v} \right)^2 + \left(\operatorname{Im} \int Y^- \bar{v} \right)^2 \right]. \quad (2.1)$$

Finally, we extend Proposition 2.2 to the operator \mathcal{L}_c linearized around a soliton $e^{ict}Q_c(x)$, by a simple scaling argument. In fact, we recall that if u is a solution of (NLS), then $w(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ is also a solution, and moreover, we have $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$ for all $c > 0$.

Corollary 2.5. *Let $c > 0$. For $v = v_1 + iv_2$, \mathcal{L}_c is defined by $\mathcal{L}_c v = -L_-v_2 + iL_+v_1$, where*

$$L_+v_1 = -\partial_x^2 v_1 + cv_1 - pQ_c^{p-1}v_1 \quad \text{and} \quad L_-v_2 = -\partial_x^2 v_2 + cv_2 - Q_c^{p-1}v_2.$$

Moreover, the spectrum $\sigma(\mathcal{L}_c)$ of \mathcal{L}_c satisfies

$$\sigma(\mathcal{L}_c) \cap \mathbb{R} = \{-e_c, 0, +e_c\}, \quad \text{where } e_c = c^{3/2}e_0 > 0.$$

Finally, e_c and $-e_c$ are simple eigenvalues of \mathcal{L}_c with eigenfunctions Y_c^+ and Y_c^- , where

$$Y_c^+(x) = c^{1/4}Y^+(\sqrt{c}x) \quad \text{and} \quad Y_c^- = \overline{Y_c^+},$$

and the null space of \mathcal{L}_c is spanned by $\partial_x Q_c$ and iQ_c .

Claim 2.6. *One can normalize Y^\pm so that*

$$-\operatorname{Im} \int (Y^+)^2 = 1 \quad \text{and still} \quad Y^- = \overline{Y^+}. \quad (2.2)$$

Proof. Denote $Y_1 = \operatorname{Re} Y^+$ and $Y_2 = \operatorname{Im} Y^+$. Thus, we have $Y^+ = Y_1 + iY_2$, $Y^- = Y_1 - iY_2$, and

$$L_+Y_1 = e_0Y_2, \quad L_-Y_2 = -e_0Y_1.$$

Now, suppose that there exists $\lambda \in \mathbb{R}$ such that $Y_2 = \lambda Q$. Then, we would have $L_-Y_2 = -e_0Y_1 = \lambda L_-Q = 0$, and so $Y_1 = 0$. But it would imply $L_+Y_1 = 0 = e_0Y_2$, and so $Y_2 = 0$, which would be a contradiction. Therefore, by (i) of Proposition 2.4, we have $\int (L_-Y_2)Y_2 = -e_0 \int Y_1Y_2 > 0$. Hence, since $\operatorname{Im} \int (Y^+)^2 = 2 \int Y_1Y_2$, we normalize Y^\pm by taking

$$\widetilde{Y^+} = \frac{Y^+}{\sqrt{-2 \int Y_1Y_2}}, \quad \widetilde{Y^-} = \overline{\widetilde{Y^+}}. \quad \square$$

2.2 Multi-solitons results

A set of parameters (1.2) being given, we adopt the following notation.

Notation 2.7. For all $j \in \llbracket 1, N \rrbracket$, define:

- (i) $\lambda_j(t, x) = x - v_j t - x_j$ and $\theta_j(t, x) = \frac{1}{2}v_j x - \frac{1}{4}v_j^2 t + c_j t + \gamma_j$.
- (ii) $R_j(t, x) = Q_{c_j}(\lambda_j(t, x))e^{i\theta_j(t, x)}$, where $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$.
- (iii) $Y_j^\pm(t, x) = Y_{c_j}^\pm(\lambda_j(t, x))e^{i\theta_j(t, x)}$, where $Y_c^\pm(x) = c^{1/4}Y^\pm(\sqrt{c}x)$.
- (iv) $e_j = e_{c_j}$, where $e_c = c^{3/2}e_0$.

Now, to estimate interactions between solitons, we denote $c_{\min} = \min\{c_k ; k \in \llbracket 1, N \rrbracket\}$, and the small parameters

$$\sigma_0 = \min\{\eta_0 \sqrt{c_{\min}}, e_0^{2/3} c_{\min}, c_{\min}, v_2 - v_1, \dots, v_N - v_{N-1}\} \quad \text{and} \quad \gamma = \frac{\sigma_0^{3/2}}{10^6}. \quad (2.3)$$

From [10], it appears that γ is a suitable parameter to quantify interactions between solitons in large time. For instance, we have, for $j \neq k$ and all $t \geq 0$,

$$\int |R_j(t)| |R_k(t)| + |(R_j)_x(t)| |(R_k)_x(t)| \leq C e^{-10\gamma t}. \quad (2.4)$$

From the definition of σ_0 and Remark 2.3, such an inequality is also true for Y_j^\pm .

Moreover, since σ_0 has the same definition as in [3], Theorem 1.2 can be rewritten as follows. *There exist $T_0 \in \mathbb{R}$, $C > 0$ and $\varphi \in C([T_0, +\infty), H^1)$ such that, for all $t \geq T_0$,*

$$\|\varphi(t) - R(t)\|_{H^1} \leq C e^{-4\gamma t}. \quad (2.5)$$

3 Construction of a family of multi-solitons

In this section, we prove Theorem 1.3 as a consequence of the following crucial Proposition 3.1. Let $p > 5$, $N \geq 2$, and a set of parameters (1.2). Denote $R = \sum_{k=1}^N R_k$ and φ a multi-soliton solution satisfying (2.5), as defined in Theorem 1.2 for example.

Proposition 3.1. *Let $j \in \llbracket 1, N \rrbracket$ and $A_j \in \mathbb{R}$. Then there exist $t_0 > 0$ and $u \in C([t_0, +\infty), H^1)$ a solution of (NLS) such that*

$$\forall t \geq t_0, \quad \|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}. \quad (3.1)$$

Before proving this proposition, let us show how this proposition implies Theorem 1.3.

Proof of Theorem 1.3. Let $(A_1, \dots, A_N) \in \mathbb{R}^N$. Denote σ the permutation of $\llbracket 1, N \rrbracket$ which satisfies

$$c_{\sigma(1)} \leq \dots \leq c_{\sigma(N)}, \quad \text{and } \sigma(i) < \sigma(j) \text{ if } c_{\sigma(i)} = c_{\sigma(j)} \text{ and } i < j.$$

- (i) Consider $\varphi_{A_{\sigma(1)}}$ the solution of (NLS) given by Proposition 3.1 applied with φ given by Theorem 1.2. Thus, there exists $t_0 > 0$ such that

$$\forall t \geq t_0, \quad \|\varphi_{A_{\sigma(1)}}(t) - \varphi(t) - A_{\sigma(1)} e^{-e_{\sigma(1)} t} Y_{\sigma(1)}^+(t)\|_{H^1} \leq e^{-(e_{\sigma(1)} + \gamma)t}.$$

Now, remark that $\varphi_{A_{\sigma(1)}}$ is also a multi-soliton which satisfies (2.5). Hence, we can apply Proposition 3.1 with $\varphi_{A_{\sigma(1)}}$ instead of φ , so that we obtain $\varphi_{A_{\sigma(1)}, A_{\sigma(2)}}$ such that

$$\forall t \geq t'_0, \quad \|\varphi_{A_{\sigma(1)}, A_{\sigma(2)}}(t) - \varphi_{A_{\sigma(1)}}(t) - A_{\sigma(2)} e^{-e_{\sigma(2)} t} Y_{\sigma(2)}^+(t)\|_{H^1} \leq e^{-(e_{\sigma(2)} + \gamma)t}.$$

Similarly, for all $j \in \llbracket 2, N \rrbracket$, we construct by induction a solution $\varphi_{A_{\sigma(1)}, \dots, A_{\sigma(j)}}$ such that

$$\forall t \geq t_0, \quad \|\varphi_{A_{\sigma(1)}, \dots, A_{\sigma(j)}}(t) - \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(j-1)}}(t) - A_{\sigma(j)} e^{-e_{\sigma(j)} t} Y_{\sigma(j)}^+(t)\|_{H^1} \leq e^{-(e_{\sigma(j)} + \gamma)t}. \quad (3.2)$$

Observe finally that $\varphi_{A_1, \dots, A_N} := \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(N)}}$ constructed by this way satisfies (2.5).

- (ii) Let $(A'_1, \dots, A'_N) \in \mathbb{R}^N$ be such that $\varphi_{A'_1, \dots, A'_N} = \varphi_{A_1, \dots, A_N}$, and let us show that it implies $(A'_1, \dots, A'_N) = (A_1, \dots, A_N)$. In fact, we prove by induction on j that $A_{\sigma(j)} = A'_{\sigma(j)}$ for

all $j \in \llbracket 1, N \rrbracket$. For $j = 1$, first note that, from the construction of $\varphi_{A_1, \dots, A_N}$, the hypothesis means $\varphi_{A'_{\sigma(1)}, \dots, A'_{\sigma(N)}} = \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(N)}}$, and moreover

$$\begin{aligned} \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(N)}}(t) &= \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(N-1)}}(t) + A_{\sigma(N)} e^{-e_{\sigma(N)} t} Y_{\sigma(N)}^+(t) + z_{\sigma(N)}(t) \\ &= \dots = \varphi(t) + \sum_{k=1}^N A_{\sigma(k)} e^{-e_{\sigma(k)} t} Y_{\sigma(k)}^+(t) + \sum_{k=1}^N z_{\sigma(k)}(t), \end{aligned}$$

where $z_{\sigma(k)}$ satisfies $\|z_{\sigma(k)}(t)\|_{H^1} \leq e^{-(e_{\sigma(k)} + \gamma)t}$ for $t \geq t_0$ and each $k \in \llbracket 1, N \rrbracket$. Similarly, we get

$$\varphi_{A'_{\sigma(1)}, \dots, A'_{\sigma(N)}}(t) = \varphi(t) + \sum_{k=1}^N A'_{\sigma(k)} e^{-e_{\sigma(k)} t} Y_{\sigma(k)}^+(t) + \sum_{k=1}^N \widetilde{z_{\sigma(k)}}(t),$$

and so, by difference, we have

$$(A_{\sigma(1)} - A'_{\sigma(1)}) e^{-e_{\sigma(1)} t} Y_{\sigma(1)}^+(t) + \sum_{k=2}^N (A_{\sigma(k)} - A'_{\sigma(k)}) e^{-e_{\sigma(k)} t} Y_{\sigma(k)}^+(t) + \sum_{k=1}^N z_{\sigma(k)}(t) - \widetilde{z_{\sigma(k)}}(t) = 0.$$

Now, if we multiply this equality by $Y_{\sigma(1)}^+(t)$, integrate, and take the imaginary part of it, we obtain, by Claim 2.6 and (2.4),

$$|A_{\sigma(1)} - A'_{\sigma(1)}| e^{-e_{\sigma(1)} t} \leq C e^{-(e_{\sigma(1)} + \gamma)t},$$

and so $A_{\sigma(1)} = A'_{\sigma(1)}$ by taking $t \rightarrow +\infty$. For the inductive step from $j-1$ to j , we write similarly

$$\begin{aligned} \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(N)}}(t) &= \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(j-1)}}(t) + \sum_{k=j}^N A_{\sigma(k)} e^{-e_{\sigma(k)} t} Y_{\sigma(k)}^+(t) + \sum_{k=j}^N z_{\sigma(k)}(t) \\ &= \varphi_{A_{\sigma(1)}, \dots, A_{\sigma(j-1)}}(t) + \sum_{k=j}^N A'_{\sigma(k)} e^{-e_{\sigma(k)} t} Y_{\sigma(k)}^+(t) + \sum_{k=j}^N \widetilde{z_{\sigma(k)}}(t), \end{aligned}$$

and we finally obtain $A_{\sigma(j)} = A'_{\sigma(j)}$ as expected, by taking the difference of these two expressions, multiplying by $Y_{\sigma(j)}^+(t)$, integrating and taking the imaginary part of it. \square

Now, the only purpose of the rest of the paper is to prove Proposition 3.1. Let $j \in \llbracket 1, N \rrbracket$ and $A_j \in \mathbb{R}$, and denote $r_j(t, x) = A_j e^{-e_j t} Y_j^+(t, x) = A_j e^{-e_j t} Y_{c_j}^+(\lambda_j(t, x)) e^{i\theta_j(t, x)}$. We want to construct a solution u of (NLS) such that

$$z(t, x) = u(t, x) - \varphi(t, x) - r_j(t, x)$$

satisfies $\|z(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}$ for $t \geq t_0$ with t_0 large enough.

3.1 Equation of z

Since u is a solution of (NLS) and also φ is (and this fact is crucial for the whole proof), we get $i\partial_t z + \partial_x^2 z + |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi|^{p-1}\varphi + A_j e^{-e_j t} e^{i\theta_j} [\partial_x^2 Y_{c_j}^+ - c_j Y_{c_j}^+ - i e_j Y_{c_j}^+](\lambda_j) = 0$. But from Corollary 2.5, we have

$$\mathcal{L}_{c_j} Y_{c_j}^+ = e_j Y_{c_j}^+ = e_j Y_{c_j,1}^+ + i e_j Y_{c_j,2}^+ = -L_- Y_{c_j,2}^+ + i L_+ Y_{c_j,1}^+$$

where $Y_{c_j,1}^+ = \operatorname{Re} Y_{c_j}^+$ and $Y_{c_j,2}^+ = \operatorname{Im} Y_{c_j}^+$, and so

$$\partial_x^2 Y_{c_j}^+ - c_j Y_{c_j}^+ + i Q_{c_j}^{p-1} Y_{c_j,2}^+ + p Q_{c_j}^{p-1} Y_{c_j,1}^+ = i e_j Y_{c_j}^+. \quad (3.3)$$

Therefore, we get the following equation for z :

$$i\partial_t z + \partial_x^2 z + |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi|^{p-1}\varphi - A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j} [pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j) = 0. \quad (3.4)$$

By developing the nonlinearity, we find

$$\begin{aligned} |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi|^{p-1}\varphi &= |\varphi + r_j|^{p-1}(\varphi + r_j) - |\varphi|^{p-1}\varphi + \omega(z) \\ &\quad + (p-1)|\varphi + r_j|^{p-3}(\varphi + r_j) \operatorname{Re}((\overline{\varphi} + \overline{r_j})z) + |\varphi + r_j|^{p-1}z, \end{aligned}$$

where $\omega(z)$ satisfies $|\omega(z)| \leq C|z|^2$ for $|z| \leq 1$. Hence, we can rewrite (3.4) as

$$i\partial_t z + \partial_x^2 z + (p-1)|\varphi + r_j|^{p-3}(\varphi + r_j) \operatorname{Re}((\overline{\varphi} + \overline{r_j})z) + |\varphi + r_j|^{p-1}z + \omega(z) = -\Omega,$$

where

$$\Omega = |\varphi + r_j|^{p-1}(\varphi + r_j) - |\varphi|^{p-1}\varphi - A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j} [pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j). \quad (3.5)$$

Finally, the equation of z can be written in the shorter form

$$i\partial_t z + \partial_x^2 z + (p-1)|\varphi|^{p-3}\varphi \operatorname{Re}(\overline{\varphi}z) + |\varphi|^{p-1}z + \omega_1 \cdot z + \omega(z) = -\Omega, \quad (3.6)$$

where ω_1 satisfies $\|\omega_1(t)\|_{L^2} \leq C e^{-e_j t}$ for all $t \geq T_0$. We finally estimate the source term Ω in the following lemma, that we prove in Appendix A.

Lemma 3.2. *There exists $C > 0$ such that, for all $t \geq T_0$, $\|\Omega(t)\|_{H^1} \leq C e^{-(e_j + 4\gamma)t}$.*

3.2 Compactness argument assuming uniform estimates

To prove Proposition 3.1, we follow the strategy of [10, 3]. We first need some notation for our purpose.

Notation 3.3. (i) Denote $J = \{k \in \llbracket 1, N \rrbracket \mid c_k \leq c_j\}$, $K = \{k \in \llbracket 1, N \rrbracket \mid c_k > c_j\}$ and $k_0 = \#K$.

(ii) \mathbb{R}^{k_0} is equipped with the ℓ^2 norm, simply denoted $\|\cdot\|$.

(iii) $\mathbb{S}_{\mathbb{R}^{k_0}}(r)$ denotes the sphere of radius r in \mathbb{R}^{k_0} .

(iv) $B_{\mathcal{B}}(r)$ is the closed ball of the Banach space \mathcal{B} , centered at the origin and of radius $r \geq 0$.

Let $S_n \rightarrow +\infty$ be an increasing sequence of time, $\mathbf{b}_n = (b_{n,k})_{k \in K} \in \mathbb{R}^{k_0}$ be a sequence of parameters to be determined, and let u_n be the solution of

$$\begin{cases} i\partial_t u_n + \partial_x^2 u_n + |u_n|^{p-1}u_n = 0, \\ u_n(S_n) = \varphi(S_n) + A_j e^{-e_j S_n} Y_j^+(S_n) + \sum_{k \in K} b_{n,k} Y_k^+(S_n). \end{cases} \quad (3.7)$$

Proposition 3.4. *There exist $n_0 \geq 0$ and $t_0 > 0$ (independent of n) such that the following holds. For each $n \geq n_0$, there exists $\mathbf{b}_n \in \mathbb{R}^{k_0}$ with $\|\mathbf{b}_n\| \leq 2e^{-(e_j + 2\gamma)S_n}$, and such that the solution u_n of (3.7) is defined on the interval $[t_0, S_n]$, and satisfies*

$$\forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}.$$

Assuming this key proposition of uniform estimates, we can sketch the proof of Proposition 3.1, relying on compactness arguments developed in [10, 3]. The proof of Proposition 3.4 is postponed to the next section.

Sketch of the proof of Proposition 3.1 assuming Proposition 3.4. From Proposition 3.4, there exists a sequence $u_n(t)$ of solutions to (NLS), defined on $[t_0, S_n]$, such that the following uniform estimates hold:

$$\forall n \geq n_0, \forall t \in [t_0, S_n], \quad \|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}.$$

In particular, there exists $C_0 > 0$ such that $\|u_n(t_0)\|_{H^1} \leq C_0$ for all $n \geq n_0$. Thus, there exists $u_0 \in H^1(\mathbb{R})$ such that $u_n(t_0) \rightharpoonup u_0$ in H^1 weak (after passing to a subsequence). Moreover, using the compactness result [10, Lemma 2], we can suppose that $u_n(t_0) \rightarrow u_0$ in L^2 strong, and so in H^{s_p} strong by interpolation, where $0 \leq s_p < 1$ is an exponent for which local well-posedness and continuous dependence hold, according to a result of Cazenave and Weissler [1]. Now, consider u solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1} u = 0, \\ u(t_0) = u_0. \end{cases}$$

Fix $t \geq t_0$. For n large enough, we have $S_n > t$, so $u_n(t)$ is defined and by continuous dependence of the solution of (NLS) upon the initial data, we have $u_n(t) \rightarrow u(t)$ in H^{s_p} strong. By the uniform H^1 bound, we also obtain $u_n(t) \rightharpoonup u(t)$ in H^1 weak. As

$$\|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t},$$

we finally obtain, by weak convergence, $\|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}$. Thus, u is a solution of (NLS) which satisfies (3.1). \square

3.3 Proof of Proposition 3.4

The proof proceeds in several steps. For the sake of simplicity, we will drop the index n for the rest of this section (except for S_n). As Proposition 3.4 is proved for given n , this should not be a source of confusion. Hence, we will write u for u_n , z for z_n , \mathbf{b} for \mathbf{b}_n , etc. We possibly drop the first terms of the sequence S_n , so that, for all n , S_n is large enough for our purposes.

From (3.6), the equation satisfied by z is

$$\begin{cases} i\partial_t z + \partial_x^2 z + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z + \omega_1 \cdot z + \omega(z) = -\Omega, \\ z(S_n) = \sum_{k \in K} b_k Y_k^+(S_n). \end{cases} \quad (3.8)$$

Moreover, for all $k \in \llbracket 1, N \rrbracket$, we denote

$$\alpha_k^\pm(t) = \operatorname{Im} \int \bar{z}(t) \cdot Y_k^\pm(t).$$

In particular, we have

$$\alpha_k^\pm(S_n) = - \sum_{l \in K} b_l \operatorname{Im} \int Y_{c_k}^\mp(\lambda_k(S_n)) Y_{c_l}^+(\lambda_l(S_n)) e^{-i\theta_k(S_n)} e^{i\theta_l(S_n)}.$$

Finally, we denote $\alpha^-(t) = (\alpha_k^-(t))_{k \in K}$.

3.3.1 Modulated final data

Lemma 3.5. *For $n \geq n_0$ large enough, the following holds. For all $\mathbf{a}^- \in \mathbb{R}^{k_0}$, there exists a unique $\mathbf{b} \in \mathbb{R}^{k_0}$ such that $\|\mathbf{b}\| \leq 2\|\mathbf{a}^-\|$ and $\alpha^-(S_n) = \mathbf{a}^-$.*

Proof. Consider the linear application

$$\begin{aligned} \Phi : \quad \mathbb{R}^{k_0} &\rightarrow \mathbb{R}^{k_0} \\ \mathbf{b} = (b_l)_{l \in K} &\mapsto (\alpha_k^-(S_n))_{k \in K}. \end{aligned}$$

If we denote $(\sigma_1, \dots, \sigma_{k_0})$ the canonical basis of \mathbb{R}^{k_0} , then, by the normalization of Claim 2.6 and the definition of Y_c^+ in Corollary 2.5, we have, for all $k \in \llbracket 1, k_0 \rrbracket$,

$$(\Phi(\sigma_k))_k = -\operatorname{Im} \int (Y_{c_k}^+)^2 = -\operatorname{Im} \int (Y^+)^2 = 1.$$

Moreover, from (2.4), there exists $C_0 > 0$ independent of n such that, for $l \neq k$,

$$|(\Phi(\sigma_k))_l| \leq \int |Y_{c_l}^+(\lambda_l(S_n))| |Y_{c_k}^+(\lambda_k(S_n))| \leq C_0 e^{-\gamma S_n}.$$

Thus, by taking n_0 large enough, we have $\Phi = \operatorname{Id} + A_n$ where $\|A_n\| \leq \frac{1}{2}$, so Φ is invertible and $\|\Phi^{-1}\| \leq 2$. Finally, for a given $\mathbf{a}^- \in \mathbb{R}^{k_0}$, it is enough to define \mathbf{b} by $\mathbf{b} = \Phi^{-1}(\mathbf{a}^-)$ to conclude the proof of Lemma 3.5. \square

Claim 3.6. *The following estimates at S_n hold:*

- $|\alpha_k^+(S_n)| \leq C e^{-2\gamma S_n} \|\mathbf{b}\|$ for all $k \in \llbracket 1, N \rrbracket$, since $\operatorname{Im} \int Y_{c_k}^- Y_{c_k}^+ = \operatorname{Im} \int |Y_{c_k}^+|^2 = 0$.
- $|\alpha_k^-(S_n)| \leq C e^{-2\gamma S_n} \|\mathbf{b}\|$ for all $k \in J$.
- $\|z(S_n)\|_{H^1} \leq C \|\mathbf{b}\|$.

3.3.2 Equations on α_k^\pm

Let $t_0 > 0$ independent of n to be determined later in the proof, $\mathbf{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ to be chosen, \mathbf{b} be given by Lemma 3.5 and u be the corresponding solution of (3.7). We now define the maximal time interval $[T(\mathbf{a}^-), S_n]$ on which suitable exponential estimates hold.

Definition 3.7. Let $T(\mathbf{a}^-)$ be the infimum of $T \geq t_0$ such that, for all $t \in [T, S_n]$, both following properties hold:

$$e^{(e_j+\gamma)t} z(t) \in B_{H^1}(1) \quad \text{and} \quad e^{(e_j+2\gamma)t} \alpha^-(t) \in B_{\mathbb{R}^{k_0}}(1). \quad (3.9)$$

Observe that Proposition 3.4 is proved if, for all n , we can find \mathbf{a}^- such that $T(\mathbf{a}^-) = t_0$. The rest of the proof is devoted to prove the existence of such a value of \mathbf{a}^- .

First, we prove the following estimate on α_k^\pm .

Claim 3.8. *For all $k \in \llbracket 1, N \rrbracket$ and all $t \in [T(\mathbf{a}^-), S_n]$,*

$$\left| \frac{d}{dt} \alpha_k^\pm(t) \mp e_k \alpha_k^\pm(t) \right| \leq C_0 e^{-4\gamma t} \|z(t)\|_{H^1} + C_1 \|z(t)\|_{H^1}^2 + C_2 e^{-(e_j+4\gamma)t}. \quad (3.10)$$

Proof. Following Notation 2.7, we compute

$$\begin{aligned} \frac{d}{dt} \alpha_k^\pm(t) &= -\frac{d}{dt} \operatorname{Im} \int \overline{Y_k^\pm(t)} z(t) = -\frac{d}{dt} \operatorname{Im} \int Y_{c_k}^\mp(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z(t) \\ &= -\operatorname{Im} \int \left[-v_k \partial_x Y_{c_k}^\mp - i(c_k - \frac{1}{4}v_k^2) Y_{c_k}^\mp \right] (x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z(t) \\ &\quad - \operatorname{Im} \int Y_{c_k}^\mp(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z_t. \end{aligned}$$

Moreover, using the equation of z (3.8) and an integration by parts, we find for the second term

$$\begin{aligned} & -\operatorname{Im} \int Y_{c_k}^\mp(x - v_k t - x_k) e^{-i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)} z_t \\ &= -\operatorname{Im} \int Y_{c_k}^\mp(\lambda_k) e^{-i\theta_k} \times i \left[\partial_x^2 z + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z + \omega_1 \cdot z + \omega(z) + \Omega \right] \end{aligned}$$

$$\begin{aligned}
&= -\operatorname{Im} \int i z e^{-i\theta_k} \left[\partial_x^2 Y_{c_k}^\mp - i v_k \partial_x Y_{c_k}^\mp - \frac{v_k^2}{4} Y_{c_k}^\mp \right] (\lambda_k) \\
&\quad - \operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} \left[(p-1) |\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z \right] \\
&\quad - \operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} [\omega_1 \cdot z + \omega(z) + \Omega].
\end{aligned}$$

Using the estimate $\|\omega_1(t)\|_{L^2} \leq C e^{-e_j t}$ and Lemma 3.2, we find for the last term

$$\left| -\operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} [\omega_1 \cdot z + \omega(z) + \Omega] \right| \leq C e^{-e_j t} \|z\|_{H^1} + C \|z\|_{H^1}^2 + C e^{-(e_j+4\gamma)t}.$$

From the definition of γ (2.3), we deduce that

$$\begin{aligned}
\frac{d}{dt} \alpha_k^\pm(t) &= -\operatorname{Im} \int i z e^{-i\theta_k} [\partial_x^2 Y_{c_k}^\mp - c_k Y_{c_k}^\mp] (\lambda_k) + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) \\
&\quad - \operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} \left[(p-1) |\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z \right].
\end{aligned}$$

Now, from (3.3), we find

$$-\operatorname{Im} \int i z e^{-i\theta_k} [\partial_x^2 Y_{c_k}^\mp - c_k Y_{c_k}^\mp] (\lambda_k) = -\operatorname{Im} \int i z e^{-i\theta_k} [\mp i e_k Y_{c_k}^\mp - i Q_{c_k}^{p-1} Y_{c_k,2}^\mp - p Q_{c_k}^{p-1} Y_{c_k,1}^\mp] (\lambda_k),$$

and, as in the proof of Lemma 3.2, we also find

$$\begin{aligned}
&-\operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} \left[(p-1) |\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + |\varphi|^{p-1} z \right] \\
&= -\operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} \left[(p-1) |R_k|^{p-3} R_k \operatorname{Re}(\overline{R_k} z) + |R_k|^{p-1} z \right] + O(e^{-4\gamma t} \|z\|_{H^1}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt} \alpha_k^\pm(t) &= \pm \left(-\operatorname{Im} \int z e^{-i\theta_k} Y_{c_k}^\mp (\lambda_k) \right) + \operatorname{Im} \int i z e^{-i\theta_k} [i Q_{c_k}^{p-1} Y_{c_k,2}^\mp + p Q_{c_k}^{p-1} Y_{c_k,1}^\mp] (\lambda_k) \\
&\quad - \operatorname{Im} \int i Y_{c_k}^\mp (\lambda_k) e^{-i\theta_k} \left[(p-1) Q_{c_k}^{p-2} (\lambda_k) e^{i\theta_k} \operatorname{Re}[Q_{c_k} (\lambda_k) e^{-i\theta_k} z] + Q_{c_k}^{p-1} (\lambda_k) z \right] \\
&\quad + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}).
\end{aligned}$$

Finally, if we denote $z_1 = \operatorname{Re}(z e^{-i\theta_k})$ and $z_2 = \operatorname{Im}(z e^{-i\theta_k})$, we find

$$\begin{aligned}
\frac{d}{dt} \alpha_k^\pm(t) &= \pm e_k \alpha_k^\pm(t) + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) \\
&\quad + \operatorname{Re} \int (z_1 + i z_2) [i Q_{c_k}^{p-1} (\lambda_k) Y_{c_k,2}^\mp (\lambda_k) + p Q_{c_k}^{p-1} (\lambda_k) Y_{c_k,1}^\mp (\lambda_k)] \\
&\quad - \operatorname{Re} \int (p-1) Y_{c_k}^\mp (\lambda_k) Q_{c_k}^{p-1} (\lambda_k) z_1 - \operatorname{Re} \int Y_{c_k}^\mp (\lambda_k) Q_{c_k}^{p-1} (\lambda_k) (z_1 + i z_2) \\
&= \pm e_k \alpha_k^\pm(t) + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) \\
&\quad + p \int z_1 Q_{c_k}^{p-1} (\lambda_k) Y_{c_k,1}^\mp (\lambda_k) - \int z_2 Q_{c_k}^{p-1} (\lambda_k) Y_{c_k,2}^\mp (\lambda_k) \\
&\quad - (p-1) \int Y_{c_k,1}^\mp (\lambda_k) Q_{c_k}^{p-1} (\lambda_k) z_1 - \int Y_{c_k,1}^\mp (\lambda_k) Q_{c_k}^{p-1} (\lambda_k) z_1 + \int Y_{c_k,2}^\mp (\lambda_k) Q_{c_k}^{p-1} (\lambda_k) z_2 \\
&= \pm e_k \alpha_k^\pm(t) + O(e^{-4\gamma t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}),
\end{aligned}$$

since all other terms cancel. \square

3.3.3 Control of the stable directions

We estimate here $\alpha_k^+(t)$ for all $k \in \llbracket 1, N \rrbracket$ and $t \in [T(\mathfrak{a}^-), S_n]$. From (3.10) and (3.9), we have

$$\left| \frac{d}{dt} \alpha_k^+(t) - e_k \alpha_k^+(t) \right| \leq C_0 e^{-(e_j+5\gamma)t} + C_1 e^{-2(e_j+\gamma)t} + C_2 e^{-(e_j+4\gamma)t} \leq K_2 e^{-(e_j+4\gamma)t}.$$

Thus, $|(e^{-e_k s} \alpha_k^+(s))'| \leq K_2 e^{-(e_j+e_k+4\gamma)s}$, and so, by integration on $[t, S_n]$, we get $|e^{-e_k S_n} \alpha_k^+(S_n) - e^{-e_k t} \alpha_k^+(t)| \leq K_2 e^{-(e_j+e_k+4\gamma)t}$, which gives

$$|\alpha_k^+(t)| \leq e^{e_k(t-S_n)} |\alpha_k^+(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But from Claim 3.6 and Lemma 3.5, we have

$$\begin{aligned} e^{e_k(t-S_n)} |\alpha_k^+(S_n)| &\leq |\alpha_k^+(S_n)| \leq C e^{-2\gamma S_n} \|\mathfrak{b}\| \\ &\leq C e^{-2\gamma S_n} e^{-(e_j+2\gamma)S_n} \leq K_2 e^{-(e_j+4\gamma)S_n} \leq K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in \llbracket 1, N \rrbracket, \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^+(t)| \leq K_2 e^{-(e_j+4\gamma)t}. \quad (3.11)$$

3.3.4 Control of the unstable directions for $k \in J$

We estimate here $\alpha_k^-(t)$ for all $k \in J$ and $t \in [T(\mathfrak{a}^-), S_n]$. Note first that, as in the previous paragraph, we get, for all $k \in \llbracket 1, N \rrbracket$ and $t \in [T(\mathfrak{a}^-), S_n]$,

$$\left| \frac{d}{dt} \alpha_k^-(t) + e_k \alpha_k^-(t) \right| \leq K_2 e^{-(e_j+4\gamma)t}. \quad (3.12)$$

Now suppose $k \in J$, which implies $e_k \leq e_j$. Since $|(e^{e_k s} \alpha_k^-(s))'| \leq K_2 e^{(e_k-e_j-4\gamma)s}$, we obtain, by integration on $[t, S_n]$,

$$|\alpha_k^-(t)| \leq e^{e_k(S_n-t)} |\alpha_k^-(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But again from Claim 3.6 and Lemma 3.5, we have

$$\begin{aligned} e^{e_k(S_n-t)} |\alpha_k^-(S_n)| &\leq K_2 e^{e_k(S_n-t)} e^{-2\gamma S_n} e^{-(e_j+2\gamma)S_n} = K_2 e^{e_k(S_n-t)} e^{-(e_j+4\gamma)S_n} \\ &\leq K_2 e^{(S_n-t)(e_k-e_j)} e^{-e_j t} e^{-4\gamma S_n} \leq K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in J, \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^-(t)| \leq K_2 e^{-(e_j+4\gamma)t}. \quad (3.13)$$

3.3.5 Localized Weinstein's functional

We follow here the same strategy as in [11, 10, 3] to estimate the energy backwards. For this, we define the function ψ by

$$\psi(x) = 0 \text{ for } x \leq -1, \quad \psi(x) = 1 \text{ for } x \geq 1, \quad \psi(x) = \frac{1}{c_0} \int_{-1}^x e^{-\frac{1}{1-y^2}} dy \quad \text{for } x \in (-1, 1),$$

where $c_0 = \int_{-1}^1 e^{-\frac{1}{1-y^2}} dy$. Hence, $\psi \in C^\infty(\mathbb{R})$ is non-decreasing and $0 \leq \psi \leq 1$. Moreover, we define, for all $k \in \llbracket 2, N \rrbracket$, $m_k(t) = \frac{1}{2} [(v_k + v_{k-1})t + x_k + x_{k-1}]$, and

$$\psi_k(t, x) = \psi \left[\frac{1}{\sqrt{t}} (x - m_k(t)) \right], \quad \psi_1 \equiv 1.$$

Moreover, we set

$$h_1(t, x) = \left(c_1 + \frac{v_1^2}{4}\right) + \sum_{k=2}^N \left[\left(c_k + \frac{v_k^2}{4}\right) - \left(c_{k-1} + \frac{v_{k-1}^2}{4}\right) \right] \psi_k(t, x),$$

$$h_2(t, x) = v_1 + \sum_{k=2}^N (v_k - v_{k-1}) \psi_k(t, x).$$

Observe that the functions h_1 and h_2 take values close to $c_k + \frac{v_k^2}{4}$ and v_k respectively, for x close to $v_k t + x_k$, and have large variations only in regions far away from the solitons. To quantify these facts (see Lemma 3.9), we introduce the functions ϕ_k , defined for $k \in \llbracket 1, N-1 \rrbracket$ by

$$\phi_k = \psi_k - \psi_{k+1}, \quad \phi_N = \psi_N.$$

Hence, we have $\phi_k \geq 0$ and $\sum_{k=1}^N \phi_k \equiv 1$, and by an Abel's transform, we also have

$$h_1 \equiv \sum_{k=1}^N \left(c_k + \frac{v_k^2}{4}\right) \phi_k \quad \text{and} \quad h_2 \equiv \sum_{k=1}^N v_k \phi_k.$$

Lemma 3.9. (i) For all $k \in \llbracket 1, N \rrbracket$, $(|R_k| + |R_{kx}|)|\phi_k - 1| \leq C e^{-4\gamma t} e^{-\sqrt{\sigma_0}|x-v_k t|}$.

(ii) For all $k, l \in \llbracket 1, N \rrbracket$ such that $l \neq k$, $(|R_k| + |R_{kx}|)\phi_l \leq C e^{-4\gamma t} e^{-\sqrt{\sigma_0}|x-v_k t|}$.

(iii) For all $k \in \llbracket 1, N \rrbracket$, $\|\phi_{kx}\|_{L^\infty} + \|\phi_{kxx}\|_{L^\infty} + \|\phi_{kt}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$.

(iv) One has $\|h_{1x}\|_{L^\infty} + \|h_{2x}\|_{L^\infty} + \|h_{1xx}\|_{L^\infty} + \|h_{2xx}\|_{L^\infty} + \|h_{1t}\|_{L^\infty} + \|h_{2t}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$, and, for all $k \in \llbracket 1, N \rrbracket$,

$$\left| h_1 - \left(c_k + \frac{v_k^2}{4}\right) \right| (|R_k| + |R_{kx}|) \leq C e^{-4\gamma t} e^{-\sqrt{\sigma_0}|x-v_k t|},$$

$$|h_2 - v_k| (|R_k| + |R_{kx}|) \leq C e^{-4\gamma t} e^{-\sqrt{\sigma_0}|x-v_k t|}.$$

Proof. See Appendix A. □

Now, we define a quantity related to the energy for z , by

$$H(t) = \int |\partial_x z|^2 - \frac{2}{p+1} \int |\varphi + r_j + z|^{p+1} - |\varphi + r_j|^{p+1} - (p+1) |\varphi + r_j|^{p-1} \operatorname{Re}[(\overline{\varphi} + \overline{r_j})z] \\ + \int h_1 |z|^2 - \operatorname{Im} \int h_2 \bar{z} \partial_x z. \quad (3.14)$$

The following estimate of the variation of H is the main new point of this paper, and as its proof is long and technical, it is postponed to Appendix B.

Proposition 3.10. For all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\left| \frac{dH}{dt}(t) \right| \leq \frac{C_0}{\sqrt{t}} \|z(t)\|_{H^1}^2 + C_1 e^{-(e_j+4\gamma)t} \|z(t)\|_{H^1} + C_2 \|z(t)\|_{H^1}^3.$$

We can now prove that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\mathcal{H}[z](t) := \int |\partial_x z|^2 - |R|^{p-1} |z|^2 - (p-1) (\operatorname{Re}(\overline{R}z))^2 |R|^{p-3} + h_1 |z|^2 - \operatorname{Im} h_2 \bar{z} \partial_x z$$

satisfies

$$\mathcal{H}[z](t) \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}. \quad (3.15)$$

Indeed, from Proposition 3.10 and estimates (3.9), we deduce that, for all $s \in [t, S_n]$,

$$\left| \frac{dH}{ds}(s) \right| \leq \frac{C_0}{\sqrt{s}} e^{-2(e_j+\gamma)s} + C_1 e^{-3\gamma s} e^{-2(e_j+\gamma)s} + C_2 e^{-3(e_j+\gamma)s} \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)s}.$$

Thus, by integration on $[t, S_n]$, we obtain $|H(t) - H(S_n)| \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}$, and so

$$H(t) \leq |H(S_n)| + \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}.$$

But from Claim 3.6 and Lemma 3.5, we have

$$\begin{aligned} |H(S_n)| &\leq C \|z(S_n)\|_{H^1}^2 \leq C \|\mathbf{b}\|^2 \leq C \|\mathbf{a}^-\|^2 \\ &\leq C e^{-2(e_j+2\gamma)S_n} \leq C e^{-2(e_j+2\gamma)t}, \end{aligned}$$

and so

$$\forall t \in [T(\mathbf{a}^-), S_n], \quad H(t) \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}.$$

Finally, expanding $|\varphi + r_j + z|^{p+1} = \left[|\varphi + r_j|^2 + 2 \operatorname{Re}[(\overline{\varphi} + \overline{r_j})z] + |z|^2 \right]^{\frac{p+1}{2}}$, we find

$$\begin{aligned} \left| |\varphi + r_j + z|^{p+1} - |\varphi + r_j|^{p+1} - (p+1) \operatorname{Re}[(\overline{\varphi} + \overline{r_j})z] |\varphi + r_j|^{p-1} - \left(\frac{p+1}{2} \right) |z|^2 |\varphi + r_j|^{p-1} \right. \\ \left. - \frac{(p+1)(p-1)}{2} (\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2 |\varphi + r_j|^{p-3} \right| \leq C |z|^3, \end{aligned}$$

and so, from the definition of H (3.14),

$$\int |\partial_x z|^2 - |\varphi + r_j|^{p-1} |z|^2 - (p-1) (\operatorname{Re}[(\overline{\varphi} + \overline{r_j})z])^2 |\varphi + r_j|^{p-3} + h_1 |z|^2 - \operatorname{Im} h_2 \bar{z} \partial_x z \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j+\gamma)t}.$$

Using (2.5), we easily obtain (3.15) by similar techniques used in the proof of Lemma 3.2 in Appendix A to replace $(\varphi + r_j)$ by R plus an exponentially small error term.

3.3.6 Control of the directions of null energy

Define $\tilde{z}(t) = z(t) + \sum_{k=1}^N \beta_k(t) i R_k(t) + \sum_{k=1}^N \gamma_k(t) \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k}$, where

$$\beta_k(t) = -\frac{\operatorname{Re} \int i R_k \bar{z}}{\|Q_{c_k}\|_{L^2}^2} = \frac{\operatorname{Im} \int R_k \bar{z}}{\|Q_{c_k}\|_{L^2}^2} \quad \text{and} \quad \gamma_k(t) = -\frac{\operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \bar{z}}{\|\partial_x Q_{c_k}\|_{L^2}^2}.$$

First, note that there exist $C_1, C_2 > 0$ such that

$$C_1 \|z\|_{H^1} \leq \|\tilde{z}\|_{H^1} + \sum_{k=1}^N (|\beta_k| + |\gamma_k|) \leq C_2 \|z\|_{H^1}. \quad (3.16)$$

Moreover, by this choice of parameters, we have, for all $k \in \llbracket 1, N \rrbracket$,

$$\left| \operatorname{Re} \int -i \overline{R_k} \tilde{z} \right| \leq C e^{-\gamma t} \|z\|_{H^1}, \quad \left| \operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \tilde{z} \right| \leq C e^{-\gamma t} \|z\|_{H^1}. \quad (3.17)$$

Indeed, by (2.4), we have

$$\begin{aligned}
\operatorname{Re} \int -i \overline{R_k} \tilde{z} &= \operatorname{Im} \int \overline{R_k} \left[z(t) + \sum_{l=1}^N \beta_l(t) i R_l(t) + \sum_{l=1}^N \gamma_l(t) \partial_x Q_{c_l}(\lambda_l) e^{i\theta_l} \right] \\
&= \operatorname{Im} \int \overline{R_k} z + \beta_k(t) \operatorname{Re} \int |R_k|^2 + \gamma_k(t) \operatorname{Im} \int Q_{c_k} \partial_x Q_{c_k} + O(e^{-\gamma t} \|z\|_{H^1}) \\
&= \operatorname{Im} \int \overline{R_k} z + \operatorname{Im} \int R_k \bar{z} + O(e^{-\gamma t} \|z\|_{H^1}) = O(e^{-\gamma t} \|z\|_{H^1}),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \bar{\tilde{z}} \\
&= \operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \bar{z} + \beta_k(t) \operatorname{Im} \int Q_{c_k} \partial_x Q_{c_k} + \gamma_k(t) \operatorname{Re} \int |\partial_x Q_{c_k}|^2 + O(e^{-\gamma t} \|z\|_{H^1}) \\
&= \operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \bar{z} - \operatorname{Re} \int \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k} \bar{z} + O(e^{-\gamma t} \|z\|_{H^1}) = O(e^{-\gamma t} \|z\|_{H^1}).
\end{aligned}$$

Now, we compare the functionals $\mathcal{H}[\tilde{z}]$ and $\mathcal{H}[z]$ in the following lemma, that we prove in Appendix A.

Lemma 3.11. *For all $t \in [T(\mathfrak{a}^-), S_n]$, one has*

$$\mathcal{H}[\tilde{z}](t) \leq \mathcal{H}[z](t) + \frac{C}{\sqrt{t}} \|z\|_{H^1}^2.$$

By (3.15) and (3.9), we deduce that

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \mathcal{H}[\tilde{z}](t) \leq \frac{K_1}{\sqrt{t}} e^{-2(e_j + \gamma)t}. \quad (3.18)$$

Now, from the property of coercivity (ii) in Proposition 2.4, and by the definitions of h_1 and h_2 , we obtain, by simple localization arguments (see [11, Appendix B] for details), that there exists $\kappa_1 > 0$ such that

$$\begin{aligned}
\mathcal{H}[\tilde{z}](t) &\geq \frac{1}{\kappa_1} \|\tilde{z}\|_{H^1}^2 - \kappa_1 \sum_{k=1}^N \left[\left(-\operatorname{Im} \int \tilde{z} \overline{Y_k^+} \right)^2 + \left(-\operatorname{Im} \int \tilde{z} \overline{Y_k^-} \right)^2 \right. \\
&\quad \left. + \left(\operatorname{Re} \int \tilde{z} (-i \overline{R_k}) \right)^2 + \left(\operatorname{Re} \int \tilde{z} \partial_x Q_{c_k}(\lambda_k) e^{-i\theta_k} \right)^2 \right]. \quad (3.19)
\end{aligned}$$

To justify heuristically this inequality, we compute, for $k \in \llbracket 1, N \rrbracket$, the localized version $\mathcal{H}_k[z]$ of $\mathcal{H}[z]$ (it would be the same for \tilde{z}), defined by

$$\mathcal{H}_k[z] = \int |\partial_x z|^2 - |R_k|^{p-1} |z|^2 - (p-1) (\operatorname{Re}(\overline{R_k} z))^2 |R_k|^{p-3} + \left(c_k + \frac{v_k^2}{4} \right) |z|^2 - v_k \operatorname{Im} \bar{z} \partial_x z.$$

In fact, if we denote $[e^{-i\theta_k} z](\cdot + v_k t + x_k) = z_1 + i z_2$, i.e. $z = e^{i\theta_k}(z_1 + i z_2)(\lambda_k)$, then we have $\partial_x z = \frac{iv_k}{2} e^{i\theta_k}(z_1 + i z_2)(\lambda_k) + e^{i\theta_k}(\partial_x z_1 + i \partial_x z_2)(\lambda_k)$, and so, by (ii) of Proposition 2.4,

$$\begin{aligned}
\mathcal{H}_k[z] &= \int \left(-\frac{v_k}{2} z_2 + \partial_x z_1 \right)^2 (\lambda_k) + \int \left(\frac{v_k}{2} z_1 + \partial_x z_2 \right)^2 (\lambda_k) \\
&\quad - \int Q_{c_k}^{p-1}(\lambda_k) (z_1^2 + z_2^2)(\lambda_k) - (p-1) \int Q_{c_k}^{p-1}(\lambda_k) z_1^2(\lambda_k) \\
&\quad + \int \left(c_k + \frac{v_k^2}{4} \right) (z_1^2 + z_2^2)(\lambda_k) - v_k \int \left(\frac{v_k}{2} z_1^2 + z_1 \partial_x z_2 + \frac{v_k}{2} z_2^2 - z_2 \partial_x z_1 \right) (\lambda_k)
\end{aligned}$$

$$\begin{aligned}
&= \int (\partial_x z_1)^2 + c_k z_1^2 - p Q_{c_k}^{p-1} z_1^2 + \int (\partial_x z_2)^2 + c_k z_2^2 - Q_{c_k}^{p-1} z_2^2 = (L_{c_k} z_1, z_1) + (L_{c_k} z_2, z_2) \\
&\geq \frac{1}{\kappa_0} \|z\|_{H^1}^2 - \kappa_0 \left[\left(\int \partial_x Q_{c_k} z_1 \right)^2 + \left(\int Q_{c_k} z_2 \right)^2 + \left(\operatorname{Im} \int Y_k^+ \bar{z} \right)^2 + \left(\operatorname{Im} \int Y_k^- \bar{z} \right)^2 \right].
\end{aligned}$$

Now, we return to (3.19), and we estimate each term of the sum, for all $k \in \llbracket 1, N \rrbracket$ and $t \in [T(\mathfrak{a}^-), S_n]$. First, by (3.17), we have

$$\left(\operatorname{Re} \int \tilde{z}(-i\overline{R_k}) \right)^2 + \left(\operatorname{Re} \int \tilde{z} \partial_x Q_{c_k}(\lambda_k) e^{-i\theta_k} \right)^2 \leq C e^{-2\gamma t} \|z\|_{H^1}^2 \leq C e^{-2\gamma t} e^{-2(e_j + \gamma)t}.$$

Second, denoting $Y_1 = \operatorname{Re} Y^+$ and $Y_2 = \operatorname{Im} Y^+$ again, we have

$$\begin{aligned}
-\operatorname{Im} \int \overline{Y_k^+}(t) \tilde{z}(t) &= \alpha_k^+(t) - \beta_k(t) \operatorname{Re} \int Q_{c_k}(\lambda_k) (Y_{c_k,1}^+ - i Y_{c_k,2}^+) (\lambda_k) \\
&\quad - \gamma_k(t) \operatorname{Im} \int \partial_x Q_{c_k}(\lambda_k) (Y_{c_k,1}^+ - i Y_{c_k,2}^+) (\lambda_k) + O(e^{-\gamma t} \|z\|_{H^1}) \\
&= \alpha_k^+(t) - C \beta_k(t) \int Q Y_1 + C \gamma_k(t) \int \partial_x Q Y_2 + O(e^{-\gamma t} \|z\|_{H^1}).
\end{aligned}$$

But by definition of Y^+ , we recall that $L_+ Y_1 = e_0 Y_2$ and $L_- Y_2 = -e_0 Y_1$, and so

$$\begin{aligned}
-\operatorname{Im} \int \overline{Y_k^+}(t) \tilde{z}(t) &= \alpha_k^+(t) + \frac{C \beta_k(t)}{e_0} \int Q(L_- Y_2) + \frac{C \gamma_k(t)}{e_0} \int \partial_x Q(L_+ Y_1) + O(e^{-\gamma t} \|z\|_{H^1}) \\
&= \alpha_k^+(t) + C' \beta_k(t) \int (L_- Q) Y_2 + C' \gamma_k(t) \int L_+ (\partial_x Q) Y_1 + O(e^{-\gamma t} \|z\|_{H^1}) \\
&= \alpha_k^+(t) + O(e^{-\gamma t} \|z\|_{H^1}),
\end{aligned}$$

since L_\pm are self-adjoint, and moreover, $L_- Q = 0$ and $L_+ (\partial_x Q) = 0$ by Proposition 2.2. Hence, by (3.11), we find, for all $k \in \llbracket 1, N \rrbracket$,

$$\left(-\operatorname{Im} \int \tilde{z} \overline{Y_k^+} \right)^2 \leq 2(\alpha_k^+)^2 + C e^{-2\gamma t} \|z\|_{H^1}^2 \leq C e^{-2(e_j + 4\gamma)t} + C e^{-2\gamma t} e^{-2(e_j + \gamma)t} \leq C e^{-2\gamma t} e^{-2(e_j + \gamma)t}.$$

Completely similarly, we find, for all $k \in \llbracket 1, N \rrbracket$,

$$\left(-\operatorname{Im} \int \tilde{z} \overline{Y_k^-} \right)^2 \leq 2(\alpha_k^-)^2 + C e^{-2\gamma t} \|z\|_{H^1}^2 \leq C e^{-2\gamma t} e^{-2(e_j + \gamma)t},$$

using (3.13) for $k \in J$, and (3.9) for $k \in K$.

Finally, gathering all estimates from (3.18), we have proved that there exists $\widetilde{K}_0 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\|\tilde{z}(t)\|_{H^1} \leq \frac{\widetilde{K}_0}{t^{1/4}} e^{-(e_j + \gamma)t}.$$

We want now to prove the same estimate for z , and so we have to control the parameters $\beta_k(t)$ and $\gamma_k(t)$ introduced above.

3.3.7 Improvement of the decay of z

Lemma 3.12. *There exists $K_0 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\|z(t)\|_{H^1} \leq \frac{K_0}{t^{1/4}} e^{-(e_j + \gamma)t}.$$

Proof. By (3.16), it is enough to prove this estimate for $|\beta_k(t)| + |\gamma_k(t)|$ with $k \in \llbracket 1, N \rrbracket$ fixed. To do this, write first the equation of \tilde{z} , from the equation of z (3.6),

$$\begin{aligned}
& i\partial_t \tilde{z} + \partial_x^2 \tilde{z} + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} \tilde{z}) + |\varphi|^{p-1} \tilde{z} \\
&= i\partial_t z - \sum \beta'_l R_l - \sum \beta_l \left[-v_l \partial_x Q_{c_l} + i \left(c_l - \frac{v_l^2}{4} \right) Q_{c_l} \right] (\lambda_l) e^{i\theta_l} + i \sum \gamma'_l \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} \\
&\quad + i \sum \gamma_l \left[-v_l \partial_x^2 Q_{c_l} + i \left(c_l - \frac{v_l^2}{4} \right) \partial_x Q_{c_l} \right] (\lambda_l) e^{i\theta_l} + \partial_x^2 z \\
&\quad + i \sum \beta_l \left[\partial_x^2 Q_{c_l} + i v_l \partial_x Q_{c_l} - \frac{v_l^2}{4} Q_{c_l} \right] (\lambda_l) e^{i\theta_l} + \sum \gamma_l \left[\partial_x^3 Q_{c_l} + i v_l \partial_x^2 Q_{c_l} - \frac{v_l^2}{4} \partial_x Q_{c_l} \right] (\lambda_l) e^{i\theta_l} \\
&\quad + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} z) + (p-1)|\varphi|^{p-3} \varphi \sum \beta_l \operatorname{Re}(i \overline{\varphi} R_l) + |\varphi|^{p-1} z \\
&\quad + (p-1)|\varphi|^{p-3} \varphi \sum \gamma_l \operatorname{Re}(\overline{\varphi} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l}) + \sum \beta_l i |\varphi|^{p-1} R_l + \sum \gamma_l |\varphi|^{p-1} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l},
\end{aligned}$$

and so, since $\partial_x^2 Q_{c_l} + Q_{c_l}^p = c_l Q_{c_l}$, we find

$$\begin{aligned}
& i\partial_t \tilde{z} + \partial_x^2 \tilde{z} + (p-1)|\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} \tilde{z}) + |\varphi|^{p-1} \tilde{z} \\
&= -\omega_1 \cdot z - \omega(z) - \Omega - \sum \beta'_l R_l + i \sum \gamma'_l \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} \\
&\quad - i \sum \beta_l Q_{c_l}^p (\lambda_l) e^{i\theta_l} - p \sum \gamma_l \partial_x Q_{c_l} (\lambda_l) Q_{c_l}^{p-1} (\lambda_l) e^{i\theta_l} \\
&\quad - (p-1) \sum \beta_l |\varphi|^{p-3} \varphi \operatorname{Im}(\overline{\varphi} R_l) + (p-1) \sum \gamma_l |\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l}) \\
&\quad + i \sum \beta_l |\varphi|^{p-1} Q_{c_l} (\lambda_l) e^{i\theta_l} + \sum \gamma_l |\varphi|^{p-1} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} \\
&= -\omega_1 \cdot z - \omega(z) - \Omega - \sum \beta'_l R_l + i \sum \gamma'_l \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} - (p-1) \sum \beta_l |\varphi|^{p-3} \varphi \operatorname{Im}(\overline{\varphi} R_l) \\
&\quad + i \sum \beta_l e^{i\theta_l} Q_{c_l} (\lambda_l) [|\varphi|^{p-1} - Q_{c_l}^{p-1} (\lambda_l)] \\
&\quad + \sum \gamma_l \left[|\varphi|^{p-1} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} + (p-1) |\varphi|^{p-3} \varphi \operatorname{Re}(\overline{\varphi} \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l}) - p \partial_x Q_{c_l} (\lambda_l) Q_{c_l}^{p-1} (\lambda_l) e^{i\theta_l} \right].
\end{aligned}$$

Then, multiply this equation by $\overline{R_k}$, integrate, and take the real part of it, so that we obtain, by (2.4), (2.5) and Lemma 3.2,

$$\begin{aligned}
& -\operatorname{Im} \int \partial_t \tilde{z} \overline{R_k} + O(\|\tilde{z}\|_{L^2}) = O(e^{-e_j t} \|z\|_{H^1}) + O(\|z\|_{H^1}^2) + O(e^{-(e_j+4\gamma)t}) - C\beta'_k \\
& \quad + \sum_{l \neq k} (\beta'_l + \gamma'_l) O(e^{-\gamma t}) + \sum \beta_l O(e^{-\gamma t}) + \sum \gamma_l O(e^{-\gamma t}).
\end{aligned}$$

In other words, we have, by (3.16) and (3.9),

$$|\beta'_k| \leq C \left| \operatorname{Im} \int \partial_t \tilde{z} \overline{R_k} \right| + C e^{-\gamma t} \sum_{l \neq k} (|\beta'_l| + |\gamma'_l|) + \frac{C}{t^{1/4}} e^{-(e_j+\gamma)t}.$$

Moreover, from

$$\operatorname{Im} \int \tilde{z} \overline{R_k} = \sum_{l \neq k} \beta_l \operatorname{Im} \int i R_l \overline{R_k} + \sum_{l \neq k} \gamma_l \operatorname{Im} \int \partial_x Q_{c_l} (\lambda_l) e^{i\theta_l} \overline{R_k},$$

we deduce that

$$\begin{aligned}
\frac{d}{dt} \operatorname{Im} \int \tilde{z} \overline{R_k} &= \sum_{l \neq k} (\beta'_l + \gamma'_l) O(e^{-\gamma t}) + \sum_{l \neq k} (\beta_l + \gamma_l) O(e^{-\gamma t}) \\
&= \operatorname{Im} \int \partial_t \tilde{z} \overline{R_k} + \operatorname{Im} \int \tilde{z} \partial_t \overline{R_k},
\end{aligned}$$

and so, as $\partial_t R_k = -v_k \partial_x R_k + i \left(c_k + \frac{v_k^2}{4} \right) R_k$,

$$\left| \operatorname{Im} \int \partial_t \tilde{z} \overline{R_k} \right| \leq C \|\tilde{z}\|_{H^1} + C e^{-\gamma t} \sum_{l \neq k} (|\beta'_l| + |\gamma'_l|) + C e^{-\gamma t} \sum_{l \neq k} (|\beta_l| + |\gamma_l|).$$

Gathering previous estimates, we find

$$|\beta'_k| \leq C e^{-\gamma t} \sum_{l \neq k} (|\beta'_l| + |\gamma'_l|) + \frac{C}{t^{1/4}} e^{-(e_j + \gamma)t}.$$

Completely similarly, if we multiply the equation on \tilde{z} by $\partial_x Q_{c_k}(\lambda_k) e^{-i\theta_k}$, integrate and take the imaginary part of it, we find

$$|\gamma'_k| \leq C e^{-\gamma t} \sum_{l \neq k} (|\beta'_l| + |\gamma'_l|) + \frac{C}{t^{1/4}} e^{-(e_j + \gamma)t}.$$

Hence, we have proved that there exist $C_3, C_4 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$|\beta'_k| + |\gamma'_k| \leq C_3 e^{-\gamma t} \sum_{l \neq k} (|\beta'_l| + |\gamma'_l|) + \frac{C_4}{t^{1/4}} e^{-(e_j + \gamma)t}.$$

Finally, if we choose t_0 large enough so that $C_3 e^{-\gamma t_0} \leq \frac{1}{N}$, we obtain, for all $s \in [t, S_n]$, with $t \in [T(\mathfrak{a}^-), S_n]$,

$$|\beta'_k(s)| + |\gamma'_k(s)| \leq \frac{C}{t^{1/4}} e^{-(e_j + \gamma)s}.$$

By integration on $[t, S_n]$, we get $|\beta_k(t)| + |\gamma_k(t)| \leq |\beta_k(S_n)| + |\gamma_k(S_n)| + \frac{C}{t^{1/4}} e^{-(e_j + \gamma)t}$. But from Claim 3.6, Lemma 3.5 and (3.16), we have

$$|\beta_k(S_n)| + |\gamma_k(S_n)| \leq C \|z(S_n)\|_{H^1} \leq C \|\mathfrak{b}\| \leq C \|\mathfrak{a}^-\| \leq C e^{-(e_j + 2\gamma)S_n} \leq C e^{-(e_j + 2\gamma)t},$$

and so finally,

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad |\beta_k(t)| + |\gamma_k(t)| \leq \frac{C}{t^{1/4}} e^{-(e_j + \gamma)t}. \quad \square$$

3.3.8 Control of the unstable directions for $k \in K$ by a topological argument

Lemma 3.12 being proved, we choose t_0 large enough so that $\frac{K_0}{t_0^{1/4}} \leq \frac{1}{2}$. Therefore, we have

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \|z(t)\|_{H^1} \leq \frac{1}{2} e^{-(e_j + \gamma)t}.$$

We can now prove the following final lemma, which concludes the proof of Proposition 3.4. Note that its proof is very similar to the one in [2], by the common choice of notation, but it is reproduced here for the reader's convenience.

Lemma 3.13. *For t_0 large enough, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j + 2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$.*

Proof. For the sake of contradiction, suppose that, for all $\mathfrak{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j + 2\gamma)S_n})$, $T(\mathfrak{a}^-) > t_0$. As $e^{(e_j + \gamma)T(\mathfrak{a}^-)} z(T(\mathfrak{a}^-)) \in B_{H^1}(1/2)$, then, by definition of $T(\mathfrak{a}^-)$ and continuity of the flow, we have

$$e^{(e_j + 2\gamma)T(\mathfrak{a}^-)} \mathfrak{a}^-(T(\mathfrak{a}^-)) \in \mathbb{S}_{\mathbb{R}^{k_0}}(1). \quad (3.20)$$

Now, let $T \in [t_0, T(\mathfrak{a}^-)]$ be close enough to $T(\mathfrak{a}^-)$ such that z is defined on $[T, S_n]$, and by continuity,

$$\forall t \in [T, S_n], \quad \|z(t)\|_{H^1} \leq e^{-(e_j + \gamma)t}.$$

We can now consider, for $t \in [T, S_n]$,

$$\mathcal{N}(t) = \mathcal{N}(\alpha^-(t)) = \|e^{(e_j+2\gamma)t}\alpha^-(t)\|^2.$$

To calculate \mathcal{N}' , we start from estimate (3.12):

$$\forall k \in K, \forall t \in [T, S_n], \quad \left| \frac{d}{dt} \alpha_k^-(t) + e_k \alpha_k^-(t) \right| \leq K'_2 e^{-(e_j+4\gamma)t}.$$

Multiplying by $|\alpha_k^-(t)|$, we obtain

$$\left| \alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + e_k \alpha_k^-(t)^2 \right| \leq K'_2 e^{-(e_j+4\gamma)t} |\alpha_k^-(t)|,$$

and thus

$$2\alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + 2e_{\min} \alpha_k^-(t)^2 \leq 2\alpha_k^-(t) \frac{d}{dt} \alpha_k^-(t) + 2e_k \alpha_k^-(t)^2 \leq K'_2 e^{-(e_j+4\gamma)t} |\alpha_k^-(t)|,$$

where $e_{\min} = \min\{e_k ; k \in K\}$. By summing on $k \in K$, we get

$$(\|\alpha^-(t)\|^2)' + 2e_{\min} \|\alpha^-(t)\|^2 \leq K_3 e^{-(e_j+4\gamma)t} \|\alpha^-(t)\|.$$

Therefore, we can estimate

$$\begin{aligned} \mathcal{N}'(t) &= (e^{2(e_j+2\gamma)t} \|\alpha^-(t)\|^2)' = e^{2(e_j+2\gamma)t} \left[2(e_j+2\gamma) \|\alpha^-(t)\|^2 + (\|\alpha^-(t)\|^2)' \right] \\ &\leq e^{2(e_j+2\gamma)t} \left[2(e_j+2\gamma) \|\alpha^-(t)\|^2 - 2e_{\min} \|\alpha^-(t)\|^2 + K_3 e^{-(e_j+4\gamma)t} \|\alpha^-(t)\| \right]. \end{aligned}$$

Hence, we have, for all $t \in [T, S_n]$,

$$\mathcal{N}'(t) \leq -\theta \cdot \mathcal{N}(t) + K_3 e^{e_j t} \|\alpha^-(t)\|,$$

where $\theta = 2(e_{\min} - e_j - 2\gamma) > 0$ by the definitions of γ (2.3) and of the set K . In particular, for all $\tau \in [T, S_n]$ satisfying $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leq -\theta + K_3 e^{e_j \tau} \|\alpha^-(\tau)\| = -\theta + K_3 e^{e_j \tau} e^{-(e_j+2\gamma)\tau} = -\theta + K_3 e^{-2\gamma\tau} \leq -\theta + K_3 e^{-2\gamma t_0}.$$

Now, we definitely fix t_0 large enough so that $K_3 e^{-2\gamma t_0} \leq \frac{\theta}{2}$, and so, for all $\tau \in [T, S_n]$ such that $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leq -\frac{\theta}{2}. \quad (3.21)$$

In particular, by (3.20), we have $\mathcal{N}'(T(\mathbf{a}^-)) \leq -\frac{\theta}{2}$.

First consequence: $\mathbf{a}^- \mapsto T(\mathbf{a}^-)$ is continuous. Indeed, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\mathcal{N}(T(\mathbf{a}^-) - \varepsilon) > 1 + \delta$ and $\mathcal{N}(T(\mathbf{a}^-) + \varepsilon) < 1 - \delta$. Moreover, by definition of $T(\mathbf{a}^-)$ and (3.21), there can not exist $\tau \in [T(\mathbf{a}^-) + \varepsilon, S_n]$ such that $\mathcal{N}(\tau) = 1$, and so by choosing δ small enough, we have, for all $t \in [T(\mathbf{a}^-) + \varepsilon, S_n]$, $\mathcal{N}(t) < 1 - \delta$. But from continuity of the flow, there exists $\eta > 0$ such that, for all $\tilde{\mathbf{a}}^-$ satisfying $\|\tilde{\mathbf{a}}^- - \mathbf{a}^-\| \leq \eta$, we have

$$\forall t \in [T(\mathbf{a}^-) - \varepsilon, S_n], \quad |\mathcal{N}(\tilde{\alpha}^-(t)) - \mathcal{N}(\alpha^-(t))| \leq \delta/2.$$

We finally deduce that $T(\mathbf{a}^-) - \varepsilon \leq T(\tilde{\mathbf{a}}^-) \leq T(\mathbf{a}^-) + \varepsilon$, as expected.

Second consequence: We can define the map

$$\begin{aligned} \mathcal{M} : B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n}) &\rightarrow \mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n}) \\ \mathbf{a}^- &\mapsto e^{-(e_j+2\gamma)(S_n-T(\mathbf{a}^-))} \alpha^-(T(\mathbf{a}^-)). \end{aligned}$$

Note that \mathcal{M} is continuous by the previous point. Moreover, let $\mathbf{a}^- \in \mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$. As $\mathcal{N}'(S_n) \leq -\frac{\theta}{2}$ by (3.21), we deduce by definition of $T(\mathbf{a}^-)$ that $T(\mathbf{a}^-) = S_n$, and so $\mathcal{M}(\mathbf{a}^-) = \mathbf{a}^-$. In other words, \mathcal{M} restricted to $\mathbb{S}_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ is the identity. But the existence of such a map \mathcal{M} contradicts Brouwer's fixed point theorem.

In conclusion, there exists $\mathbf{a}^- \in B_{\mathbb{R}^{k_0}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathbf{a}^-) = t_0$. \square

A Appendix

Proof of Lemma 3.2. First, we calculate

$$\begin{aligned}
|R_j|^{p-1}r_j + (p-1)|R_j|^{p-3}R_j \operatorname{Re}(\overline{R_j}r_j) &= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) [Y_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j) e^{i\theta_j} \\
&\quad + (p-1)Q_{c_j}^{p-2}(\lambda_j) e^{i\theta_j} \operatorname{Re}[A_j e^{-e_j t} Q_{c_j}(Y_{c_j,1}^+ + iY_{c_j,2}^+)](\lambda_j) \\
&= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j} [Y_{c_j,1}^+ + iY_{c_j,2}^+ + (p-1)Y_{c_j,1}^+](\lambda_j) \\
&= A_j e^{-e_j t} Q_{c_j}^{p-1}(\lambda_j) e^{i\theta_j} [pY_{c_j,1}^+ + iY_{c_j,2}^+](\lambda_j).
\end{aligned}$$

Hence, from the expression of Ω (3.5), it can be written

$$\Omega = |\varphi + r_j|^{p-1}(\varphi + r_j) - |\varphi|^{p-1}\varphi - |R_j|^{p-1}r_j - (p-1)|R_j|^{p-3}R_j \operatorname{Re}(\overline{R_j}r_j).$$

We can now estimate $\|\Omega\|_{H^1}$, and we estimate $\|\partial_x \Omega\|_{L^2}$ for example, the term $\|\Omega\|_{L^2}$ being similar and easier. To do this, we write

$$\begin{aligned}
\Omega_x &= (p-1) \operatorname{Re}[(\varphi_x + r_{jx})(\overline{\varphi} + \overline{r_j})] |\varphi + r_j|^{p-3}(\varphi + r_j) + |\varphi + r_j|^{p-1}(\varphi_x + r_{jx}) \\
&\quad - (p-1) \operatorname{Re}(\varphi_x \overline{\varphi}) |\varphi|^{p-3}\varphi - |\varphi|^{p-1}\varphi_x - (p-1) \operatorname{Re}(R_{jx} \overline{R_j}) |R_j|^{p-3}r_j - |R_j|^{p-1}r_{jx} \\
&\quad - (p-1)(p-3) \operatorname{Re}(R_{jx} \overline{R_j}) |R_j|^{p-5}R_j \operatorname{Re}(\overline{R_j}r_j) - (p-1)|R_j|^{p-3}R_{jx} \operatorname{Re}(\overline{R_j}r_j) \\
&\quad - (p-1)|R_j|^{p-3}R_j \operatorname{Re}(\overline{R_{jx}}r_j) - (p-1)|R_j|^{p-3}R_j \operatorname{Re}(\overline{R_j}r_{jx}) \\
&= (p-1) \operatorname{Re}(\varphi_x \overline{\varphi}) \left[|\varphi + r_j|^{p-3}(\varphi + r_j) - |\varphi|^{p-3}\varphi - (p-3)\varphi \operatorname{Re}(\overline{\varphi}r_j) |\varphi|^{p-5} - |\varphi|^{p-3}r_j \right] \\
&\quad + (p-1)(p-3) \left[\operatorname{Re}(\varphi_x \overline{\varphi}) \operatorname{Re}(\overline{\varphi}r_j) |\varphi|^{p-5}\varphi - \operatorname{Re}(R_{jx} \overline{R_j}) \operatorname{Re}(\overline{R_j}r_j) |R_j|^{p-5}R_j \right] \\
&\quad + (p-1)r_j \left[\operatorname{Re}(\varphi_x \overline{\varphi}) |\varphi|^{p-3} - \operatorname{Re}(R_{jx} \overline{R_j}) |R_j|^{p-3} \right] \\
&\quad + (p-1) \left[\operatorname{Re}(\varphi_x \overline{r_j}) |\varphi + r_j|^{p-3}(\varphi + r_j) - \operatorname{Re}(\overline{R_{jx}}r_j) |R_j|^{p-3}R_j \right] \\
&\quad + (p-1) \left[\operatorname{Re}(r_{jx} \overline{\varphi}) |\varphi + r_j|^{p-3}(\varphi + r_j) - \operatorname{Re}(r_{jx} \overline{R_j}) |R_j|^{p-3}R_j \right] \\
&\quad + (p-1) \operatorname{Re}(r_{jx} \overline{r_j}) |\varphi + r_j|^{p-3}(\varphi + r_j) + r_{jx} \left[|\varphi + r_j|^{p-1} - |R_j|^{p-1} \right] \\
&\quad + \varphi_x \left[|\varphi + r_j|^{p-1} - |\varphi|^{p-1} - (p-1) \operatorname{Re}(\overline{\varphi}r_j) |\varphi|^{p-3} \right] \\
&\quad + (p-1) \left[\operatorname{Re}(\overline{\varphi}r_j) \varphi_x |\varphi|^{p-3} - \operatorname{Re}(\overline{R_j}r_j) R_{jx} |R_j|^{p-3} \right].
\end{aligned}$$

To estimate all these terms in L^2 norm, we use the facts that φ is equal to R plus a small error term according to (2.5), that R multiplied by a term moving on the line $x = v_j t + x_j$ (like r_j) is equal to R_j plus a small error term according to (2.4), and finally that r_j is at order $e^{-e_j t}$. To illustrate this, we estimate the first two terms **I** and **II**, for example, as all other terms can be treated similarly. For **I**, we simply remark that

$$\|\mathbf{I}\|_{L^2} \leq C \|r_j\|_{L^2}^2 \leq C e^{-2e_j t} \leq C e^{-(e_j + 4\gamma)t}$$

by the definition of γ (2.3). For **II**, we decompose it as

$$\begin{aligned}
\frac{1}{(p-1)(p-3)} \mathbf{II} &= \operatorname{Re}[(\varphi_x - R_x) \overline{\varphi}] \operatorname{Re}(\overline{\varphi}r_j) |\varphi|^{p-5}\varphi + \operatorname{Re}(R_x(\overline{\varphi} - \overline{R})) \operatorname{Re}(\overline{\varphi}r_j) |\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}(R_x \overline{R}) \operatorname{Re}[(\overline{\varphi} - \overline{R})r_j] |\varphi|^{p-5}\varphi + \operatorname{Re}[(R_x - R_{jx}) \overline{R}] \operatorname{Re}(\overline{R}r_j) |\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}[R_{jx}(\overline{R} - \overline{R_j})] \operatorname{Re}(\overline{R}r_j) |\varphi|^{p-5}\varphi + \operatorname{Re}[R_{jx} \overline{R_j}] \operatorname{Re}[(\overline{R} - \overline{R_j})r_j] |\varphi|^{p-5}\varphi \\
&\quad + \operatorname{Re}(R_{jx} \overline{R_j}) \operatorname{Re}(\overline{R_j}r_j) \left[|\varphi|^{p-5}\varphi - |R_j|^{p-5}R_j \right].
\end{aligned}$$

Since $\|\varphi - R\|_{H^1} \leq Ce^{-4\gamma t}$ by (2.5), the first three terms are bounded in L^2 norm by $Ce^{-(e_j+4\gamma)t}$. Moreover, by (2.4), the next three terms are also bounded in L^2 norm by $Ce^{-(e_j+4\gamma)t}$. Finally, for the last term, we write

$$|\varphi|^{p-5}\varphi - |R_j|^{p-5}R_j = (|\varphi|^{p-5}\varphi - |R|^{p-5}R) + (|R|^{p-5}R - |R_j|^{p-5}R_j),$$

so that, since $p > 5$, we can conclude similarly that $\|\mathbf{II}\|_{L^2} \leq Ce^{-(e_j+4\gamma)t}$. \square

Proof of Lemma 3.9. (i) For $k \in \llbracket 1, N \rrbracket$, we have

$$\begin{aligned} (|R_k| + |R_{kx}|)|\phi_k - 1| &\leq Ce^{-\sqrt{c_k}|x-v_k t|}[1 + \psi_{k+1} - \psi_k] \\ &\leq Ce^{-\sqrt{\sigma_0}|x-v_k t|} \cdot e^{-\sqrt{\sigma_0}|x-v_k t|}[1 + \psi_{k+1} - \psi_k]. \end{aligned}$$

But, if $x < m_k(t) + \sqrt{t}$, then

$$e^{-\sqrt{\sigma_0}|x-v_k t|}[1 + \psi_{k+1} - \psi_k] \leq Ce^{\sqrt{\sigma_0}x}e^{-\sqrt{\sigma_0}v_k t} \leq Ce^{\frac{1}{2}\sqrt{\sigma_0}(v_k+v_{k-1}-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leq Ce^{-\frac{1}{4}\sigma_0^{3/2}t},$$

and similarly, if $x > m_{k+1}(t) - \sqrt{t}$, then

$$e^{-\sqrt{\sigma_0}|x-v_k t|}[1 + \psi_{k+1} - \psi_k] \leq Ce^{-\sqrt{\sigma_0}x}e^{\sqrt{\sigma_0}v_k t} \leq Ce^{-\frac{1}{2}\sqrt{\sigma_0}(v_{k+1}-v_k-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leq Ce^{-\frac{1}{4}\sigma_0^{3/2}t}.$$

As $\phi_k(t, x) = 1$ for $m_k(t) + \sqrt{t} \leq x \leq m_{k+1}(t) - \sqrt{t}$, the conclusion follows from (2.3).

(ii) For $l, k \in \llbracket 1, N \rrbracket$ such that $l \neq k$, we have

$$\begin{aligned} (|R_k| + |R_{kx}|)\phi_l &\leq Ce^{-\sqrt{c_k}|x-v_k t|}[\psi_l - \psi_{l+1}]\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}} \\ &\leq Ce^{-\sqrt{\sigma_0}|x-v_k t|} \cdot e^{-\sqrt{\sigma_0}|x-v_k t|}\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}}. \end{aligned}$$

But, if $k > l$, then

$$\begin{aligned} e^{-\sqrt{\sigma_0}|x-v_k t|}\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}} &\leq e^{\sqrt{\sigma_0}x}e^{-\sqrt{\sigma_0}v_k t}\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}} \\ &\leq Ce^{\frac{1}{2}\sqrt{\sigma_0}(v_{l+1}+v_l-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leq Ce^{-\frac{1}{4}\sigma_0^{3/2}t}, \end{aligned}$$

and similarly, if $k < l$, then

$$\begin{aligned} e^{-\sqrt{\sigma_0}|x-v_k t|}\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}} &\leq Ce^{-\sqrt{\sigma_0}x}e^{\sqrt{\sigma_0}v_k t}\mathbb{1}_{\{x > m_l(t) - \sqrt{t}\}}\mathbb{1}_{\{x < m_{l+1}(t) + \sqrt{t}\}} \\ &\leq Ce^{-\frac{1}{2}\sqrt{\sigma_0}(v_l+v_{l-1}-2v_k)t}e^{\sqrt{\sigma_0}\sqrt{t}} \leq Ce^{-\frac{1}{4}\sigma_0^{3/2}t}, \end{aligned}$$

and the conclusion follows again from the definition of γ .

(iii) For $k \in \llbracket 1, N \rrbracket$, it suffices to prove $\|\psi_{kx}\|_{L^\infty} + \|\psi_{kxx}\|_{L^\infty} + \|\psi_{kt}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$. The first two inequalities are obvious since $\psi_{kx}(t, x) = \frac{1}{\sqrt{t}}\psi' \left[\frac{1}{\sqrt{t}}(x - m_k(t)) \right]$ and so $\|\psi_{kx}\|_{L^\infty} \leq \frac{1}{\sqrt{t}}\|\psi'\|_{L^\infty}$, and similarly $\|\psi_{kxx}\|_{L^\infty} \leq \frac{1}{t}\|\psi''\|_{L^\infty}$. For the last one, we write

$$\psi_k(t, x) = \psi \left[\frac{x - \frac{1}{2}(x_k + x_{k-1})}{\sqrt{t}} - \frac{1}{2}(v_k + v_{k-1})\sqrt{t} \right],$$

so that

$$\psi_{kt}(t, x) = \left[-\frac{1}{2} \left(\frac{x - \frac{x_k + x_{k-1}}{2}}{t^{3/2}} \right) - \frac{1}{4} \left(\frac{v_k + v_{k-1}}{\sqrt{t}} \right) \right] \cdot \psi' \left[\frac{1}{\sqrt{t}}(x - m_k(t)) \right] \mathbb{1}_{|x - m_k(t)| \leq \sqrt{t}},$$

since $\text{supp}(\psi') = [-1, 1]$. But for x such that $|x - m_k(t)| \leq \sqrt{t}$, we have $\left| x - \frac{x_k + x_{k-1}}{2} \right| \leq Ct$, and so finally $\|\psi_{kt}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}\|\psi'\|_{L^\infty}$.

- (iv) Since $h_1 \equiv \sum_{k=1}^N \left(c_k + \frac{v_k^2}{4}\right) \phi_k$ and $h_2 \equiv \sum_{k=1}^N v_k \phi_k$ have a similar form, it is clear that it suffices to prove the inequalities for h_2 , for example. Moreover, the first inequalities are obvious by (iii). Finally, for the last inequality, we write

$$\begin{aligned} |h_2 - v_k|(|R_k| + |R_{kx}|) &= \left| \sum_{l=1}^N v_l \phi_l - v_k \right| (|R_k| + |R_{kx}|) \\ &\leq v_k |\phi_k - 1| (|R_k| + |R_{kx}|) + \sum_{l \neq k} v_l \phi_l (|R_k| + |R_{kx}|) \leq C e^{-4\gamma t} e^{-\sqrt{\sigma_0}|x-v_k t|} \end{aligned}$$

by (i) and (ii), which concludes the proof. \square

Proof of Lemma 3.11. To compare $\mathcal{H}[\tilde{z}]$ and $\mathcal{H}[z]$, we replace \tilde{z} in $\mathcal{H}[\tilde{z}]$ by its definition,

$$\tilde{z} = z + \sum_{k=1}^N \beta_k i Q_{c_k}(\lambda_k) e^{i\theta_k} + \sum_{k=1}^N \gamma_k \partial_x Q_{c_k}(\lambda_k) e^{i\theta_k},$$

dropping the argument λ_k for this proof, which would not be a source of confusion since there is no time derivative. Hence, we compute

$$\begin{aligned} \mathcal{H}[\tilde{z}] &= \int \partial_x \tilde{z} \cdot \overline{\partial_x \tilde{z}} - \text{Im } h_2 \partial_x \tilde{z} \cdot \tilde{z} + (h_1 - |R|^{p-1}) \tilde{z} \cdot \tilde{z} - (p-1) (\text{Re}(\overline{R} \tilde{z}))^2 |R|^{p-3} \\ &= \int \left[\partial_x z + \sum \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} + i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{i\theta_k} \right] \\ &\quad \times \left[\partial_x \bar{z} + \sum \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} - i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{-i\theta_k} \right] \\ &\quad - \int h_2 \text{Im} \left[\partial_x z + \sum \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} + i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{i\theta_k} \right] \\ &\quad \times \left[\bar{z} + \sum (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) e^{-i\theta_k} \right] \\ &\quad + \int (h_1 - |R|)^{p-1} \left[z + \sum (\gamma_k \partial_x Q_{c_k} + i \beta_k Q_{c_k}) e^{i\theta_k} \right] \times \left[\bar{z} + \sum (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) e^{-i\theta_k} \right] \\ &\quad - \int (p-1) |R|^{p-3} \left[\text{Re}(\overline{R} z) - \sum \beta_k \text{Im}(R_k \overline{R}) + \sum \gamma_k \text{Re}(\partial_x Q_{c_k} e^{i\theta_k} \overline{R}) \right]^2. \end{aligned}$$

Developing in terms of z , we find

$$\begin{aligned} \mathcal{H}[\tilde{z}] &= \int |\partial_x z|^2 + 2 \text{Re} \int \partial_x z \cdot \sum \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} - i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{-i\theta_k} \\ &\quad + \sum_{k,l} \int \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} + i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{i\theta_k} \\ &\quad \quad \times \left(\gamma_l \partial_x^2 Q_{c_l} - \frac{\beta_l}{2} v_l Q_{c_l} - i \partial_x Q_{c_l} (\beta_l + \frac{1}{2} v_l \gamma_l) \right) e^{-i\theta_l} \\ &\quad - \text{Im} \int h_2 \partial_x z \cdot \bar{z} - \text{Im} \int h_2 \partial_x z \cdot \sum (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) e^{-i\theta_k} \\ &\quad + \text{Im} \int h_2 z \cdot \sum \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} - i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{-i\theta_k} \\ &\quad - \sum_{k,l} \text{Im} \int h_2 \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} + i \partial_x Q_{c_k} (\beta_k + \frac{1}{2} v_k \gamma_k) \right) e^{i\theta_k} (\gamma_l \partial_x Q_{c_l} - i \beta_l Q_{c_l}) e^{-i\theta_l} \\ &\quad + \int (h_1 - |R|^{p-1}) |z|^2 + 2 \text{Re} \int (h_1 - |R|^{p-1}) z \cdot \sum (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) e^{-i\theta_k} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k,l} \int (h_1 - |R|^{p-1}) (\gamma_k \partial_x Q_{c_k} + i \beta_k Q_{c_k}) e^{i\theta_k} (\gamma_l \partial_x Q_{c_l} - i \beta_l Q_{c_l}) e^{-i\theta_l} \\
& - (p-1) \int |R|^{p-3} (\operatorname{Re}(\overline{R}z))^2 - (p-1) \int |R|^{p-3} \sum_{k,l} \beta_k \beta_l \operatorname{Im}(R_k \overline{R}) \operatorname{Im}(R_l \overline{R}) \\
& - (p-1) \int |R|^{p-3} \sum_{k,l} \gamma_k \gamma_l \operatorname{Re}(\partial_x Q_{c_k} e^{i\theta_k} \overline{R}) \operatorname{Re}(\partial_x Q_{c_l} e^{i\theta_l} \overline{R}) \\
& + 2(p-1) \int |R|^{p-3} \operatorname{Re}(\overline{R}z) \sum \beta_k \operatorname{Im}(R_k \overline{R}) \\
& - 2(p-1) \int |R|^{p-3} \operatorname{Re}(\overline{R}z) \sum \gamma_k \operatorname{Re}(\partial_x Q_{c_k} e^{i\theta_k} \overline{R}) \\
& + 2(p-1) \int |R|^{p-3} \sum_{k,l} \beta_k \gamma_l \operatorname{Im}(R_k \overline{R}) \operatorname{Re}(\partial_x Q_{c_l} e^{i\theta_l} \overline{R}).
\end{aligned}$$

Now, first remark that $\text{Im}(R_k \overline{R}) = \sum_{q \neq k} \text{Im}(R_k \overline{R}_q)$, and so, by (2.4), all integrals containing this term are in $O(e^{-\gamma t} \|z\|_{H^1}^2)$. Moreover, still by (2.4), all double sums on k, l have their terms in $O(e^{-\gamma t} \|z\|_{H^1}^2)$ whenever $k \neq l$. Note finally that all terms composing $\mathcal{H}[z]$ appear. Hence, with an integration by parts to make $\partial_x z$ disappear, we have

$$\begin{aligned}
H[\tilde{z}] = & \int |\partial_x z|^2 - \text{Im } h_2 \partial_x z \cdot \bar{z} + (h_1 - |R|^{p-1})|z|^2 - (p-1)|R|^{p-3}(\text{Re}(\bar{R}z))^2 + O(e^{-\gamma t} \|z\|_{H^1}^2) \\
& - 2 \sum \text{Re} \int z e^{-i\theta_k} \left[\left(\gamma_k \partial_x^3 Q_{c_k} - \beta_k v_k \partial_x Q_{c_k} - \frac{1}{4} \gamma_k v_k^2 \partial_x Q_{c_k} \right) \right. \\
& \quad \left. + i \left(-v_k \gamma_k \partial_x^2 Q_{c_k} - \beta_k \partial_x^2 Q_{c_k} + \frac{1}{4} v_k^2 \beta_k Q_{c_k} \right) \right] \\
& + \sum \int \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} \right)^2 + \left(\beta_k + \frac{1}{2} v_k \gamma_k \right)^2 (\partial_x Q_{c_k})^2 \\
& + \sum \text{Im} \int z \partial_x h_2 (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) e^{-i\theta_k} \\
& + 2 \sum \text{Im} \int h_2 z e^{-i\theta_k} \left[\left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} \right) - i \partial_x Q_{c_k} \left(\beta_k + \frac{1}{2} v_k \gamma_k \right) \right] \\
& - \sum \int h_2 \gamma_k \left(\beta_k + \frac{1}{2} v_k \gamma_k \right) (\partial_x Q_{c_k})^2 + \sum \int h_2 \beta_k Q_{c_k} \left(\gamma_k \partial_x^2 Q_{c_k} - \frac{\beta_k}{2} v_k Q_{c_k} \right) \\
& + 2 \sum \text{Re} \int (h_1 - |R|^{p-1}) z e^{-i\theta_k} (\gamma_k \partial_x Q_{c_k} - i \beta_k Q_{c_k}) \\
& + \sum \int (h_1 - |R|^{p-1}) (\gamma_k^2 (\partial_x Q_{c_k})^2 + \beta_k^2 Q_{c_k}^2) \\
& - (p-1) \sum \int |R|^{p-3} \gamma_k^2 Q_{c_k}^2 (\partial_x Q_{c_k})^2 - 2(p-1) \sum \text{Re} \int |R|^{p-3} z e^{-i\theta_k} \gamma_k Q_{c_k}^2 \partial_x Q_{c_k}.
\end{aligned}$$

We now use notation $z_{1,k} = \operatorname{Re}(z^{-i\theta_k})$ and $z_{2,k} = \operatorname{Im}(z^{-i\theta_k})$ again. Moreover, recall that we have $\|\partial_x h_2\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$ by (iv) of Lemma 3.9, and $\partial_x^2 Q_{c_k} + Q_{c_k}^p = c_k Q_{c_k}$ by (1.1). Thus, we find

$$\begin{aligned} \mathcal{H}[\tilde{z}] &= \mathcal{H}[z] + O(t^{-1/2} \|z\|_{H^1}^2) \\ &+ \sum \int z_{1,k} [-2c_k \gamma_k \partial_x Q_{c_k} + 2p \gamma_k \partial_x Q_{c_k} Q_{c_k}^{p-1} + 2\beta_k v_k \partial_x Q_{c_k} + \frac{1}{2} \gamma_k v_k^2 \partial_x Q_{c_k} \\ &- 2h_2 \beta_k \partial_x Q_{c_k} - h_2 \gamma_k v_k \partial_x Q_{c_k} + 2h_1 \gamma_k \partial_x Q_{c_k} - 2\gamma_k \partial_x Q_{c_k} Q_{c_k}^{p-1} - 2(p-1) \gamma_k \partial_x Q_{c_k} Q_{c_k}^{p-1}] \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned}
& + \sum \int z_{2,k} [-2\gamma_k v_k c_k Q_{c_k} + 2\gamma_k v_k Q_{c_k}^p - 2\beta_k c_k Q_{c_k} + 2\beta_k Q_{c_k}^p + \frac{1}{2}\beta_k v_k^2 Q_{c_k} + 2h_2 \gamma_k c_k Q_{c_k} \\
& \quad - 2h_2 \gamma_k Q_{c_k}^p - h_2 \beta_k v_k Q_{c_k} + 2h_1 \beta_k Q_{c_k} - 2\beta_k Q_{c_k}^p] \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
& + \sum \int \left(\gamma_k c_k Q_{c_k} - \gamma_k Q_{c_k}^p - \frac{\beta_k}{2} v_k Q_{c_k} \right)^2 + \left(\beta_k + \frac{1}{2} v_k \gamma_k \right)^2 (\partial_x Q_{c_k})^2 \\
& - \sum \int h_2 \gamma_k (\beta_k + \frac{1}{2} v_k \gamma_k) (\partial_x Q_{c_k})^2 + \sum \int h_2 \beta_k Q_{c_k} (\gamma_k c_k Q_{c_k} - \gamma_k Q_{c_k}^p - \frac{\beta_k}{2} v_k Q_{c_k}) \\
& + \sum \int h_1 [\gamma_k^2 (\partial_x Q_{c_k})^2 + \beta_k^2 Q_{c_k}^2] - \sum \int Q_{c_k}^{p-1} [\gamma_k^2 (\partial_x Q_{c_k})^2 + \beta_k^2 Q_{c_k}^2] \\
& - \sum \int (p-1) \gamma_k^2 Q_{c_k}^{p-1} (\partial_x Q_{c_k})^2. \tag{A.3}
\end{aligned}$$

To conclude, we estimate the term (A.1) involving $z_{1,k}$, the term (A.2) involving $z_{2,k}$, and finally the source term (A.3). For (A.1), we write

$$(A.1) = \sum \int z_{1,k} \gamma_k \partial_x Q_{c_k} (-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1) + 2 \sum \int z_{1,k} \beta_k \partial_x Q_{c_k} (v_k - h_2),$$

and $-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1 = 2(h_1 - c_k - \frac{v_k^2}{4}) + v_k(v_k - h_2)$, so that, by (iv) of Lemma 3.9, we have $(A.1) = O(e^{-\gamma t} \|z\|_{H^1}^2)$. Similarly, we write

$$\begin{aligned}
(A.2) & = 2 \sum \int z_{2,k} \gamma_k c_k Q_{c_k} (h_2 - v_k) + 2 \sum \int z_{2,k} \gamma_k Q_{c_k}^p (v_k - h_2) \\
& \quad + \sum \int z_{2,k} \beta_k Q_{c_k} (-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1),
\end{aligned}$$

and we also conclude that $(A.2) = O(e^{-\gamma t} \|z\|_{H^1}^2)$. For the last term, we expand it as

$$\begin{aligned}
(A.3) & = \sum \int \beta_k \gamma_k c_k Q_{c_k}^2 (h_2 - v_k) + \beta_k \gamma_k Q_{c_k}^{p+1} (v_k - h_2) + \beta_k \gamma_k (\partial_x Q_{c_k})^2 (v_k - h_2) \\
& + \sum \int \gamma_k^2 c_k^2 Q_{c_k}^2 + \gamma_k^2 Q_{c_k}^{2p} + \frac{\beta_k^2}{4} v_k^2 Q_{c_k}^2 - 2\gamma_k^2 c_k Q_{c_k}^{p+1} + \beta_k^2 (\partial_x Q_{c_k})^2 + \frac{1}{4} \gamma_k^2 v_k^2 (\partial_x Q_{c_k})^2 \\
& - \frac{1}{2} h_2 \gamma_k^2 v_k (\partial_x Q_{c_k})^2 - \frac{1}{2} h_2 \beta_k^2 v_k Q_{c_k}^2 + h_1 \gamma_k^2 (\partial_x Q_{c_k})^2 + h_1 \beta_k^2 Q_{c_k}^2 - \beta_k^2 Q_{c_k}^{p+1} - p \gamma_k^2 Q_{c_k}^{p-1} (\partial_x Q_{c_k})^2.
\end{aligned}$$

Note that the first sum is in $O(e^{-\gamma t} \|z\|_{H^1}^2)$ as above. Hence, with several integrations by parts and using $\partial_x^2 Q_{c_k} = c_k Q_{c_k} - Q_{c_k}^p$, we find

$$\begin{aligned}
(A.3) & = O(e^{-\gamma t} \|z\|_{H^1}^2) + \sum \int \gamma_k^2 c_k^2 Q_{c_k}^2 + \gamma_k^2 Q_{c_k}^{2p} + \frac{\beta_k^2}{4} v_k^2 Q_{c_k}^2 - 2\gamma_k^2 c_k Q_{c_k}^{p+1} - \beta_k^2 Q_{c_k} (c_k Q_{c_k} - Q_{c_k}^p) \\
& \quad - \frac{1}{4} \gamma_k^2 v_k^2 Q_{c_k} (c_k Q_{c_k} - Q_{c_k}^p) + \frac{1}{2} h_2 \gamma_k^2 v_k Q_{c_k} (c_k Q_{c_k} - Q_{c_k}^p) - \frac{1}{2} h_2 \beta_k^2 v_k Q_{c_k}^2 \\
& \quad - h_1 \gamma_k^2 Q_{c_k} (c_k Q_{c_k} - Q_{c_k}^p) + h_1 \beta_k^2 Q_{c_k}^2 - \beta_k^2 Q_{c_k}^{p+1} + \gamma_k^2 Q_{c_k}^p (c_k Q_{c_k} - Q_{c_k}^p) \\
& = O(e^{-\gamma t} \|z\|_{H^1}^2) - \frac{1}{2} \sum \int \gamma_k^2 c_k Q_{c_k}^2 (-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1) \\
& \quad + \frac{1}{2} \sum \int \beta_k^2 Q_{c_k}^2 (-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1) + \frac{1}{2} \sum \int \gamma_k^2 Q_{c_k}^{p+1} (-2c_k + \frac{v_k^2}{2} - h_2 v_k + 2h_1),
\end{aligned}$$

and so we can conclude as above that $(A.3) = O(e^{-\gamma t} \|z\|_{H^1}^2)$. Finally, we proved that $\mathcal{H}[\tilde{z}] = \mathcal{H}[z] + O(t^{-1/2} \|z\|_{H^1}^2)$, as expected. \square

B Appendix

We prove here Proposition 3.10. To do this, we first need a lemma quantifying the fact that φ almost satisfies a transport equation similar to those satisfied by the solitons. Note finally that, since φ_t takes values in H^{-1} , all integrals in this appendix may be seen as the dual bracket $\langle \cdot, \cdot \rangle_{H^1, H^{-1}}$.

Lemma B.1. *There exists $C > 0$ such that, for all $t \geq T_0$, $\|\varphi_t + h_2\varphi_x - ih_1\varphi\|_{H^{-1}} \leq Ce^{-4\gamma t}$.*

Remark B.2. To find the transport equation almost satisfied by φ , it suffices to compute an exact relation for R_k with $k \in \llbracket 1, N \rrbracket$. In fact, as

$$R_k(t, x) = Q_{c_k}(x - v_k t - x_k) e^{i(\frac{1}{2}v_k x - \frac{1}{4}v_k^2 t + c_k t + \gamma_k)},$$

we have $R_{kt} = [-v_k \partial_x Q_{c_k} + i(c_k - \frac{1}{4}v_k^2)Q_{c_k}](\lambda_k) e^{i\theta_k}$ and $R_{kx} = [\partial_x Q_{c_k} + \frac{i}{2}v_k Q_{c_k}](\lambda_k) e^{i\theta_k}$, and so

$$R_{kt} + v_k R_{kx} - i \left(c_k + \frac{v_k^2}{4} \right) R_k = 0.$$

Proof of Lemma B.1. Let $f \in H^1$ and compute

$$\begin{aligned} \int (\varphi_t + h_2\varphi_x - ih_1\varphi)f &= \int (i\varphi_{xx} + i|\varphi|^{p-1}\varphi + h_2\varphi_x - ih_1\varphi)f \\ &= i \int (\varphi_{xx} - R_{xx})f + i \int (|\varphi|^{p-1}\varphi - |R|^{p-1}R)f + \int h_2(\varphi_x - R_x)f - i \int h_1(\varphi - R)f \\ &\quad + i \int (R_{xx} + |R|^{p-1}R - ih_2R_x - h_1R)f \\ &= -i \int (\varphi_x - R_x)f_x + i \int (|\varphi|^{p-1}\varphi - |R|^{p-1}R)f + \int h_2(\varphi_x - R_x)f - i \int h_1(\varphi - R)f \\ &\quad + i \sum_{k=1}^N \int (R_{kxx} + |R_k|^{p-1}R_k - ih_2R_{kx} - h_1R_k)f + i \sum_{k=1}^N \int R_k(|R|^{p-1} - |R_k|^{p-1})f \\ &= \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned}$$

First note that, by (2.5), $|\mathbf{I}| \leq C\|\varphi - R\|_{H^1}\|f\|_{H^1} \leq Ce^{-4\gamma t}\|f\|_{H^1}$. Moreover, by (2.4), we also have $|\mathbf{III}| \leq Ce^{-4\gamma t}\|f\|_{L^2}$. For the last term, we first compute

$$\begin{cases} R_k = Q_{c_k}(\lambda_k) e^{i\theta_k}, & R_{kx} = (\partial_x Q_{c_k} + \frac{i}{2}v_k Q_{c_k})(\lambda_k) e^{i\theta_k}, \\ R_{kxx} = (\partial_x^2 Q_{c_k} + iv_k \partial_x Q_{c_k} - \frac{v_k^2}{4} Q_{c_k})(\lambda_k) e^{i\theta_k}, \end{cases}$$

and so, using $\partial_x^2 Q_{c_k} = c_k Q_{c_k} - Q_{c_k}^p$, we obtain

$$\begin{aligned} \mathbf{II} &= i \sum_{k=1}^N \int \left[\left(c_k - \frac{v_k^2}{4} - h_1 \right) R_k + iv_k R_{kx} + \frac{v_k^2}{2} R_k - ih_2 R_{kx} \right] f \\ &= i \sum_{k=1}^N \int \left(c_k + \frac{v_k^2}{4} - h_1 \right) R_k f + \sum_{k=1}^N \int (h_2 - v_k) R_{kx} f. \end{aligned}$$

Therefore, by (iv) of Lemma 3.9, we also have $|\mathbf{II}| \leq Ce^{-4\gamma t}\|f\|_{L^2}$, which concludes the proof of Lemma B.1. \square

Proof of Proposition 3.10. First recall that, from Section 3.1, the equation of z can be written

$$iz_t + z_{xx} + |\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) = -\Omega,$$

where $r_j(t, x) = A_j e^{-e_j t} Y_j^+(t, x)$ and Ω satisfies $\|\Omega\|_{H^1} \leq Ce^{-(e_j + 4\gamma)t}$ by Lemma 3.2.

From the definition of H (3.14), we now compute, using integrations by parts,

$$\begin{aligned}
H'(t) &= 2 \operatorname{Re} \int z_{tx} \bar{z}_x - 2 \operatorname{Re} \int (\varphi + r_j + z)_t |\varphi + r_j + z|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) \\
&\quad + 2 \operatorname{Re} \int (\varphi + r_j)_t |\varphi + r_j|^{p-1} (\bar{\varphi} + \bar{r}_j) \\
&\quad + 2(p-1) \operatorname{Re} \int (\varphi + r_j)_t |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z] \\
&\quad + 2 \int |\varphi + r_j|^{p-1} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)_t z] + 2 \int |\varphi + r_j|^{p-1} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z_t] \\
&\quad + \int h_{1t} |z|^2 + 2 \operatorname{Re} \int h_{1t} z_t \bar{z} - \operatorname{Im} \int h_{2t} z_x \bar{z} - \operatorname{Im} \int h_{2t} z_x \bar{z} - \operatorname{Im} \int h_{2t} z_x \bar{z} \\
&= -2 \operatorname{Re} \int z_t [\bar{z}_{xx} + |\varphi + r_j + z|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) - |\varphi + r_j|^{p-1} (\bar{\varphi} + \bar{r}_j)] \\
&\quad - 2 \operatorname{Re} \int (\varphi + r_j)_t [|\varphi + r_j + z|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) \\
&\quad \quad - |\varphi + r_j|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) - (p-1) |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z]] \\
&\quad + 2 \operatorname{Re} \int h_{1t} z_t \bar{z} + 2 \operatorname{Im} \int h_{2t} \bar{z}_x z_t + \operatorname{Im} \int h_{2t} z_x \bar{z} + \int h_{1t} |z|^2 - \operatorname{Im} \int h_{2t} z_x \bar{z}.
\end{aligned}$$

But from (iv) of Lemma 3.9, we have $\|h_{1t}\|_{L^\infty} + \|h_{2t}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$, and so

$$\left| \int h_{1t} |z|^2 - \operatorname{Im} \int h_{2t} z_x \bar{z} \right| \leq \frac{C}{\sqrt{t}} \|z\|_{H^1}^2.$$

Moreover, by expanding $|\varphi + r_j + z|^{p-1} = [|\varphi + r_j + z|^2]^{\frac{p-1}{2}}$, and as $\|r_{jt}\|_{L^\infty} \leq C e^{-e_j t}$, we have

$$\begin{aligned}
&\left| -2 \operatorname{Re} \int r_{jt} [|\varphi + r_j + z|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) - |\varphi + r_j|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) \right. \\
&\quad \left. - (p-1) |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z] \right] \right| \leq C e^{-4\gamma t} \|z\|_{H^1}^2.
\end{aligned}$$

Hence, replacing z_t by its equation, we find

$$\begin{aligned}
H'(t) &= -2 \operatorname{Im} \int \bar{\Omega} [z_{xx} + |\varphi + r_j + z|^{p-1} (\varphi + r_j + z) - |\varphi + r_j|^{p-1} (\varphi + r_j)] \\
&\quad - 2 \operatorname{Re} \int \varphi_t [|\varphi + r_j + z|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) \\
&\quad \quad - |\varphi + r_j|^{p-1} (\bar{\varphi} + \bar{r}_j + \bar{z}) - (p-1) |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z]] \\
&\quad - 2 \operatorname{Im} \int h_{1t} \bar{z} z_{xx} - 2 \operatorname{Im} \int h_{1t} \bar{z} [|\varphi + r_j + z|^{p-1} (\varphi + r_j + z) - |\varphi + r_j|^{p-1} (\varphi + r_j)] \\
&\quad - 2 \operatorname{Im} \int h_{1t} \Omega \bar{z} + 2 \operatorname{Re} \int h_{2t} \bar{z}_x z_{xx} + \operatorname{Re} \int h_{2t} \bar{z}_x z_{xx} + \operatorname{Re} \int (2h_{2t} \bar{z}_x + h_{2t} \bar{z}) \Omega \\
&\quad - 2 \operatorname{Re} \int h_{2t} \bar{z} [|\varphi + r_j + z|^{p-1} (\varphi + r_j + z) - |\varphi + r_j|^{p-1} (\varphi + r_j)]_x \\
&\quad - \operatorname{Re} \int h_{2t} \bar{z} [|\varphi + r_j + z|^{p-1} (\varphi + r_j + z) - |\varphi + r_j|^{p-1} (\varphi + r_j)] + O(t^{-1/2} \|z\|_{H^1}^2).
\end{aligned}$$

We can already estimate several terms in this expression. For the first term, for example, we have, by an integration by parts,

$$\left| -2 \operatorname{Im} \int \bar{\Omega} z_{xx} \right| = \left| 2 \operatorname{Im} \int \bar{\Omega}_x z_x \right| \leq C \|\Omega\|_{H^1} \|z\|_{H^1} \leq C e^{-(e_j + 4\gamma)t} \|z\|_{H^1}.$$

Similarly, we have

$$\begin{aligned} \left| -2 \operatorname{Im} \int \bar{\Omega} \left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) \right] \right| &\leq C e^{-(e_j+4\gamma)t} \|z\|_{H^1}, \\ \left| -2 \operatorname{Im} \int h_1 \Omega \bar{z} + \operatorname{Re} \int (2h_2 \bar{z}_x + h_{2x} \bar{z}) \Omega \right| &\leq C e^{-(e_j+4\gamma)t} \|z\|_{H^1}. \end{aligned}$$

Then, another integration by parts gives

$$-2 \operatorname{Im} \int h_1 \bar{z} z_{xx} = 2 \operatorname{Im} \int h_1 |z_x|^2 + 2 \operatorname{Im} \int h_{1x} \bar{z} z_x = 2 \operatorname{Im} \int h_{1x} \bar{z} z_x,$$

and so, as $\|h_{1x}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$ by Lemma 3.9, $|-2 \operatorname{Im} \int h_1 \bar{z} z_{xx}| \leq \frac{C}{\sqrt{t}} \|z\|_{H^1}^2$. As we also have $\|h_{2x}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$, we can estimate

$$\left| -\operatorname{Re} \int h_{2x} \bar{z} \left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) \right] \right| \leq \frac{C}{\sqrt{t}} \|z\|_{H^1}^2.$$

Finally, we can also estimate

$$\begin{aligned} 2 \operatorname{Re} \int h_2 \bar{z}_x z_{xx} + \operatorname{Re} \int h_{2x} \bar{z} z_{xx} &= - \int h_{2x} |z_x|^2 - \operatorname{Re} \int z_x (h_{2xx} \bar{z} + h_{2x} \bar{z}_x) \\ &= -2 \int h_{2x} |z_x|^2 - \operatorname{Re} \int h_{2xx} z_x \bar{z}. \end{aligned}$$

Indeed, since $\|h_{2x}\|_{L^\infty} + \|h_{2xx}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$ by Lemma 3.9, we have

$$\left| 2 \operatorname{Re} \int h_2 \bar{z}_x z_{xx} + \operatorname{Re} \int h_{2x} \bar{z} z_{xx} \right| \leq \frac{C}{\sqrt{t}} \|z\|_{H^1}^2.$$

Gathering all previous estimates, we have proved that

$$-\frac{1}{2} H'(t) = \mathbf{I} + \mathbf{II} + \mathbf{III} + O(e^{-(e_j+4\gamma)t} \|z\|_{H^1}) + O(t^{-1/2} \|z\|_{H^1}^2),$$

where

$$\begin{cases} \mathbf{I} = \operatorname{Re} \int h_2 \bar{z} \left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) \right]_x, \\ \mathbf{II} = \operatorname{Im} \int h_1 \bar{z} \left[|\varphi + r_j + z|^{p-1}(\varphi + r_j + z) - |\varphi + r_j|^{p-1}(\varphi + r_j) \right], \\ \mathbf{III} = \operatorname{Re} \int \varphi_t \left[|\varphi + r_j + z|^{p-1}(\bar{\varphi} + \bar{r}_j + \bar{z}) - |\varphi + r_j|^{p-1}(\bar{\varphi} + \bar{r}_j + \bar{z}) \right. \\ \quad \left. - (p-1) |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] \right]. \end{cases}$$

The purpose is now to make appear quadratic terms in z in these expressions. For \mathbf{II} and \mathbf{III} , we simply write

$$\mathbf{II} = -\operatorname{Re} \int i h_1 \bar{z} \left[|\varphi + r_j|^{p-1} z + (p-1)(\varphi + r_j) |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] \right] + O(\|z\|_{H^1}^3)$$

and

$$\begin{aligned} \mathbf{III} = \operatorname{Re} \int \varphi_t &\left[\left(\frac{p-1}{2} \right) |z|^2 |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j) + (p-1) \bar{z} |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] \right. \\ &\quad \left. + \frac{(p-1)(p-3)}{2} (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z])^2 |\varphi + r_j|^{p-5} (\bar{\varphi} + \bar{r}_j) \right] + O(\|z\|_{H^1}^3). \end{aligned}$$

For **I**, we have to compute

$$\begin{aligned}
\mathbf{I} &= \operatorname{Re} \int \bar{z} h_2 \left\{ (p-1) |\varphi + r_j + z|^{p-3} \operatorname{Re}[(\varphi + r_j + z)_x (\bar{\varphi} + \bar{r}_j + \bar{z})] (\varphi + r_j + z) \right. \\
&\quad + |\varphi + r_j + z|^{p-1} (\varphi + r_j + z)_x - (p-1) |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] (\varphi + r_j) \\
&\quad \left. - |\varphi + r_j|^{p-1} (\varphi + r_j)_x \right\} \\
&= \operatorname{Re} \int \bar{z} h_2 \left\{ (p-1) \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] (\varphi + r_j) \left[|\varphi + r_j + z|^{p-3} - |\varphi + r_j|^{p-3} \right] \right. \\
&\quad + (p-1) z |\varphi + r_j + z|^{p-3} \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] + (\varphi + r_j)_x \bar{z} + z_x (\bar{\varphi} + \bar{r}_j) + z_x \bar{z} \\
&\quad + (p-1) (\varphi + r_j) |\varphi + r_j + z|^{p-3} \operatorname{Re}[(\varphi + r_j)_x \bar{z} + z_x (\bar{\varphi} + \bar{r}_j) + z_x \bar{z}] \\
&\quad \left. + (\varphi + r_j)_x \left[|\varphi + r_j + z|^{p-1} - |\varphi + r_j|^{p-1} \right] + |\varphi + r_j + z|^{p-1} z_x \right\} \\
&= \operatorname{Re} \int \bar{z} h_2 \left\{ (p-1)(p-3) |\varphi + r_j|^{p-5} (\varphi + r_j) \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] \right. \\
&\quad + (p-1) z |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] \\
&\quad + (p-1) (\varphi + r_j) |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x \bar{z} + z_x (\bar{\varphi} + \bar{r}_j)] \\
&\quad \left. + (p-1) (\varphi + r_j)_x |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] + |\varphi + r_j|^{p-1} z_x \right\} + O(\|z\|_{H^1}^3).
\end{aligned}$$

In the last expression, we integrate by parts the following two terms. First, we have

$$\begin{aligned}
&\operatorname{Re} \int \bar{z} h_2 \cdot (p-1) (\varphi + r_j) |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x \bar{z} + z_x (\bar{\varphi} + \bar{r}_j)] \\
&= (p-1) \int \operatorname{Re}[z (\bar{\varphi} + \bar{r}_j)] \operatorname{Re}[z (\bar{\varphi} + \bar{r}_j)]_x h_2 |\varphi + r_j|^{p-3} \\
&= - \left(\frac{p-1}{2} \right) \int (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z])^2 h_{2x} |\varphi + r_j|^{p-3} \\
&\quad - \frac{(p-1)(p-3)}{2} \int (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z])^2 |\varphi + r_j|^{p-5} h_2 \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)].
\end{aligned}$$

Second, we have similarly

$$\begin{aligned}
\operatorname{Re} \int \bar{z} h_2 z_x |\varphi + r_j|^{p-1} &= - \frac{1}{2} \int |z|^2 \left[h_{2x} |\varphi + r_j|^{p-1} + h_2 (p-1) |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] \right] \\
&= - \frac{1}{2} \int |z|^2 h_{2x} |\varphi + r_j|^{p-1} - \left(\frac{p-1}{2} \right) \operatorname{Re} \int h_2 (\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j) |\varphi + r_j|^{p-3} |z|^2.
\end{aligned}$$

Therefore, as $\|h_{2x}\|_{L^\infty} \leq \frac{C}{\sqrt{t}}$, we have obtained

$$\begin{aligned}
-\frac{1}{2} H'(t) &= O(e^{-(e_j+4\gamma)t} \|z\|_{H^1}) + O(t^{-1/2} \|z\|_{H^1}^2) + O(\|z\|_{H^1}^3) \\
&\quad + \frac{(p-1)(p-3)}{2} \int (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z])^2 |\varphi + r_j|^{p-5} h_2 \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] \\
&\quad + \left(\frac{p-1}{2} \right) \int h_2 |z|^2 |\varphi + r_j|^{p-3} \operatorname{Re}[(\varphi + r_j)_x (\bar{\varphi} + \bar{r}_j)] \\
&\quad + (p-1) \operatorname{Re} \int \bar{z} h_2 (\varphi + r_j)_x |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z] \\
&\quad + \frac{(p-1)(p-3)}{2} \operatorname{Re} \int \varphi_t (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j) z])^2 |\varphi + r_j|^{p-5} (\bar{\varphi} + \bar{r}_j) \\
&\quad + \left(\frac{p-1}{2} \right) \operatorname{Re} \int \varphi_t |z|^2 |\varphi + r_j|^{p-3} (\bar{\varphi} + \bar{r}_j)
\end{aligned}$$

$$\begin{aligned}
& + (p-1) \operatorname{Re} \int \varphi_t \bar{z} |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z] \\
& - (p-1) \operatorname{Re} \int i h_1 \bar{z} (\varphi + r_j) |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z].
\end{aligned}$$

Finally, collecting similar terms in a single integral, we get, as $\|r_j\|_{H^1} \leq C e^{-e_j t}$,

$$\begin{aligned}
-\frac{1}{2} H'(t) &= O(e^{-(e_j+4\gamma)t} \|z\|_{H^1}) + O(t^{-1/2} \|z\|_{H^1}^2) + O(\|z\|_{H^1}^3) \\
&+ \frac{(p-1)(p-3)}{2} \operatorname{Re} \int \bar{\varphi} |\varphi + r_j|^{p-5} (\operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z])^2 [\varphi_t + h_2 \varphi_x - i h_1 \varphi] \\
&+ \left(\frac{p-1}{2}\right) \operatorname{Re} \int |z|^2 \bar{\varphi} |\varphi + r_j|^{p-3} [\varphi_t + h_2 \varphi_x - i h_1 \varphi] \\
&+ (p-1) \operatorname{Re} \int \bar{z} |\varphi + r_j|^{p-3} \operatorname{Re}[(\bar{\varphi} + \bar{r}_j)z] [\varphi_t + h_2 \varphi_x - i h_1 \varphi] \\
&= O(e^{-(e_j+4\gamma)t} \|z\|_{H^1}) + O(t^{-1/2} \|z\|_{H^1}^2) + O(\|z\|_{H^1}^3),
\end{aligned}$$

since $\|\varphi_t + h_2 \varphi_x - i h_1 \varphi\|_{H^{-1}} \leq C e^{-4\gamma t}$ by Lemma B.1 and the three terms in front of $\varphi_t + h_2 \varphi_x - i h_1 \varphi$ are bounded in H^1 by $\|z\|_{H^1}^2$, which concludes the proof of Proposition 3.10. \square

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